Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

M3M6/M4M6

Advanced techniques for solving integral and differential equations

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The function $y(x)$ satisfies the nonlinear second–order differential equation

$$\epsilon y'' + y' - \tan y = 0 \ , \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1 \ ,$$

with boundary conditions $y(0) = 0$ and $y(1) = \pi/2$.

(i) Carefully describe the scaling (i.e. set $x = x_0 + \epsilon^\alpha X$ and find α). By analysing the lowest–order inner solution, locate the boundary layer (i.e. find x_0). Write down the inner equation. Decide which boundary condition must be satisfied by the outer solution.

(ii) Find the leading (lowest–order) outer and inner approximations to $y(x)$ and match them. Sketch the solution, indicating the inner and outer regions.

2. The function $y(x)$ satisfies the linear second–order differential equation

$$\epsilon y'' - 3y' + y = 0 \ , \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1 \ ,$$

with boundary conditions $y(0) = 1$ and $y(1) = 0$.

(i) Determine the scaling and the inner equation.

(ii) Obtain the two-term outer expansion and the two-term inner expansion of the function $y(x)$ and match the two expansions.

3. The function $y(x)$ satisfies the integral equation

$$\frac{dy}{dx} + 6y(x) = 12 \int_0^\infty e^{-4|x-t|} y(t) dt \ , \quad 0 \le x < \infty \ ,$$

and the boundary condition $y(0) = 2$.

(i) Find $y(x)$ using the Wiener-Hopf method.

(ii) Sketch $y(x)$ and find the limiting value $k = \lim_{x \to \infty} y(x)$.

[The standard result

$$\int_0^\infty e^{-\alpha x} e^{isx} dx = \frac{i}{s + i\alpha} \ , \quad \mathrm{Im} s > -\alpha \ ,$$

for real α, may be used without proof.]

4. Solve the integral equation

$$\frac{1}{\pi}\int_{-1}^{1} \ln|t-x| f(t)dt = -\ln\left(1 - x^2/4\right) \ , \quad -1 \le x \le 1 \ .$$

[The inversion formula

$$f(x) = -\frac{1}{\pi\sqrt{1-x^2}} \mathrm{P}\int_{-1}^{1} dt \frac{\sqrt{1-t^2}}{t-x} g(t) + \frac{A}{\sqrt{1-x^2}}$$

for the singular integral equation

$$\frac{1}{\pi}\mathrm{P}\int_{-1}^{1} \frac{dt}{t-x} f(t) = g(x)$$

may be quoted without proof; here P indicates the principal value integral and A is an arbitrary constant.]

5. The (generalised) Laguerre polynomials $L_n^\alpha(x)$, $x \in [0, \infty]$, satisfy the second order differential equation

$$x\frac{d^2 L_n^\alpha}{dx^2} + (\alpha + 1 - x)\frac{dL_n^\alpha}{dx} + nL_n^\alpha = 0 \ ,$$

where α is a real parameter.

(i) Bring this equation into the standard form:

$$\frac{d}{dx}\left[p(x)\frac{dL_n^\alpha}{dx}\right] + q(x)\frac{dL_n^\alpha}{dx} + nL_n^\alpha = 0 \ .$$

Hence find the functions $p(x)$ and $q(x)$ and the weight function $w(x)$.

(ii) Write down the Rodrigues formula and the orthogonality relations. What is the condition imposed on the parameter α by the convergence of the orthogonality integral? By using the Rodrigues formula, or otherwise, obtain $L_0^\alpha(x)$, $L_1^\alpha(x)$, and $L_2^\alpha(x)$ normalised in such a way that the coefficient of x^n in L_n^α is unity.

(iii) Determine the generating function $G(x, y)$ for Laguerre polynomials such that

$$G^\alpha(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} L_n^\alpha(x) y^n \ .$$

[Hint: recall from notes that $G = [w(z)/w(x)](\partial z/\partial x)$, given that z satisfies $z = x + yp(z)$.] Using these power series and the Laguerre equation above, obtain the linear homogeneous second order partial differential equation satisfied by the function $G^\alpha(x, y)$.