Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

# M3M6/M4M6

## Advanced techniques for solving integral and differential equations

Date: Wednesday, 31st May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The function $y(x)$ satisfies the nonlinear second–order differential equation

$$\epsilon y'' - 2y' + e^y = 0 \ , \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1 \ ,$$

with boundary conditions $y(0) = 0$ and $y(1) = 0$.

(i) Carefully describe the scaling (i.e. set $x = x_0 + \epsilon^\alpha X$ and find $\alpha$). By analysing the lowest–order inner solution, locate the boundary layer (i.e. find $x_0$). Write down the inner equation. Decide which boundary condition must be satisfied by the outer solution.

(ii) Find the leading (lowest–order) outer and inner approximations to $y(x)$ and match them. Sketch the solution, indicating the inner and outer regions.

2. The function $y(x)$ satisfies the linear second–order differential equation

$$\epsilon y'' + y' + xy = 0 \ , \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1 \ ,$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$.

Determine the scaling and the inner equation.

Obtain the two-term outer expansion and the two-term inner expansion of the function $y(x)$ and match the two expansions.

3. The function $y(x)$ satisfies the integral equation

$$y(x) = \frac{1}{2} \int_0^\infty e^{-|x-t|} y(t) dt + e^{-2x} \ , \quad 0 \le x < \infty \ ,$$

and the boundary condition $y(0) = 3/2$.

Find $y(x)$ using the Wiener-Hopf method. Sketch $y(x)$ and find the limiting value $k = \lim_{x \to \infty} y(x)$.

[The standard result

$$\int_0^\infty e^{-\alpha x} e^{isx} dx = \frac{i}{s + i\alpha} \ , \quad \mathrm{Im} s > -\alpha \ ,$$

for real $\alpha$, may be used without proof.]

4. Solve the integral equation

$$\frac{1}{\pi}\int_{-1}^{1} \ln|t-x| f(t)dt = -\tan^{-1} x \ , \quad -1 \le x \le 1 \ .$$

[The inversion formula

$$f(x) = -\frac{1}{\pi\sqrt{1-x^2}} \mathrm{P}\int_{-1}^{1} dt \frac{\sqrt{1-t^2}}{t-x} g(t) + \frac{A}{\sqrt{1-x^2}}$$

for the singular integral equation

$$\frac{1}{\pi}\mathrm{P}\int_{-1}^{1} \frac{dt}{t-x} f(t) = g(x)$$

may be quoted without proof; here $\mathrm{P}$ indicates the principal value integral and $A$ is an arbitrary constant.]

5. The generating function for Chebyshev polynomials of the first kind $T_n(x)$, $-1 < x < 1$, is

$$G(x,y) = \frac{1-xy}{1-2xy+y^2} = \sum_{n=0}^{\infty} T_n(x) y^n \ .$$

By differentiation, show that the generating function satisfies the following partial differential equation

$$(1-x^2)\frac{\partial^2 G}{\partial x^2} - x\frac{\partial G}{\partial x} + y\frac{\partial}{\partial y}\left(y\frac{\partial G}{\partial y}\right) = 0 \ .$$

From this, obtain the Chebyshev equation, i.e. the second–order differential equation satisfied by $T_n(x)$. Bring this equation into the standard form:

$$\frac{d}{dx}\left[p(x)\frac{dT_n}{dx}\right] + q(x)\frac{dT_n}{dx} + n^2 T_n = 0 \ .$$

Hence find the functions $p(x)$ and $q(x)$ and the weight function $w(x)$.

Write down the Rodrigues formula and the orthogonality relations. By using the Rodrigues formula, or otherwise, obtain $T_0(x)$, $T_1(x)$, and $T_2(x)$ normalised in such a way that the coefficient of $x^n$ in $T_n$ is unity.