1. The function $y(x)$ satisfies the nonlinear second–order differential equation

$$\epsilon y'' + y' + x^3 y^2 = 0\,, \quad 0 \le x \le 2, \quad 0 < \epsilon \ll 1\,,$$

with boundary conditions $y(0) = 0$ and $y(2) = 1/5$.

Carefully describe the scaling (i.e. set $x = x_0 + \epsilon^\alpha X$ and find α). By analysing the lowest–order inner solution, locate the boundary layer (i.e. find x_0). Write down the inner equation. Decide which boundary condition must be satisfied by the outer solution.

Find the leading (lowest–order) outer and inner approximations to $y(x)$ and match them. Sketch the solution, indicating the inner and outer regions.

2. The function $y(x)$ satisfies the linear second–order differential equation

$$\epsilon y'' - 2y' + 2y = 0\,, \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1\,,$$

with boundary conditions $y(0) = 1$ and $y(1) = 0$.

Determine the scaling and the inner equation.

Obtain the two-term outer expansion and the two-term inner expansion of the function $y(x)$ and match the two expansions.

3. The function $y(x)$ satisfies the integro–differential equation

$$\frac{df(x)}{dx} + f(x) = \int_0^\infty e^{-|x-y|} f(y)\,dy\,, \quad 0 \le x < \infty\,,$$

with the boundary condition $f(0) = 1$.

Find $f(x)$ using the Wiener-Hopf method.

[The standard result

$$\int_0^\infty e^{-\alpha x} e^{isx}\,dx = \frac{i}{s + i\alpha}\,, \quad \mathrm{Im} s > -\alpha\,,$$

for real α, may be used without proof.]

4. The function $f(x)$ satisfies the singular integral equation,

$$\frac{1}{\pi} \mathrm{P} \int_{-1}^{1} \frac{dt}{t-x} f(t) = g(x) \ , \quad -1 < x < 1 \ ,$$

where the function $g(x)$ is given and P indicates the principle value integral. The associated function is defined by

$$F(z) = \frac{1}{2\pi i} \mathrm{P} \int_{-1}^{1} \frac{dx}{x-z} f(x) \ ,$$

where z is a complex variable.

Using Plemelj formulae for the function $F(z)$,

$$F_{+}(x) - F_{-}(x) = f(x) \ ,$$

$$F_{+}(x) + F_{-}(x) = -ig(x) \ ,$$

derive the invesion formula, i.e. find the expression for the function $f(x)$ in terms of the function $g(x)$.

Hence obtain all the solutions to the integral equation in the case when $g(x) = 0$ for all $x \in [-1, 1]$ and verify this result by substituting it into the integral equation and evaluating the principal value integral.

[Hint: use the function $\omega(z) = (z^2 - 1)^{1/2} F(z)$ to reduce the problem to the converse jump discontinuity problem.]

5. The generating function for Laguerre polynomials $P_n(x)$, $0 < x < \infty$, is

$$G(x, y) = \frac{e^{-xy/(1-y)}}{1-y} = \sum_{n=0}^{\infty} P_n(x) y^n \ .$$

By differentiation, show that the generating function satisfies the following partial differential equation

$$x \frac{\partial^2 G}{\partial x^2} + (1-x) \frac{\partial G}{\partial x} + y \frac{\partial G}{\partial y} = 0 \ .$$

From this, obtain the Legendre equation, i.e. the second–order differential equation satisfied by $P_n(x)$. [Hint: equate powers of y.] Bring this equation into the standard form:

$$\frac{d}{dx}\left[p(x) \frac{dP_n}{dx} \right] + q(x) \frac{dP_n}{dx} + n(n+1) P_n = 0 \ .$$

Hence find the functions $p(x)$ and $q(x)$ and the weight function $w(x)$.

Write down the Rodrigues formula and the orthogonality relations. By using the Rodrigues formula, or otherwise, obtain $P_0(x)$, $P_1(x)$, and $P_2(x)$ normalised so that the coefficient of x^n in P_n is unity.