1. Consider the initial-value problem

$$x' = f(x,t), \quad x(0) = 0,$$

where x is an n-dimensional vector, and $f(x,t)$ an n-dimensional vector function defined for $x \in D: |x| \le \beta$ and $t \in I: |t| \le \alpha$.

(a) (i) State the Cauchy-Peano Theorem concerning the existence of a solution to the initial-value problem.

(ii) Suppose that $f(x,t)$ is a two-dimensional vector function

$$f(x,t) = \left(\frac{\sqrt{1-t^2}}{1+x_2^2},\ (\cos t)x_1^2\right)^T,$$

where $|x| \le \beta$ and $|t| \le 1$. In what range of t does a solution to the initial value problem exist according to the Cauchy-Peano theorem?

(b) Suppose that $f(x,t)$ is a continuous vector function of x and t, and $\dfrac{\partial f}{\partial x}$ exists and $|\dfrac{\partial f}{\partial x}| \le L$ for $|x| \le \beta$ and $|t| \le \alpha$.

(i) Prove that $f(x,t)$ satisfies a Lipschitz condition.

(ii) Use the above result, or otherwise, to show that the two-dimensional vector function

$$f(x,t) = \left((x_1^2 + x_2^2)^\gamma,\ x_1 x_2\right)^T,$$

where $|x| \le \beta$ and $\gamma \ge \frac{1}{2}$ is a parameter, satisfies a Lipschitz condition.

If $0 < \gamma < \frac{1}{2}$, would $f(x,t)$ satisfy a Lipschitz condition? Why?

(c) (i) For the $f(x,t)$ given in (b), state the range of γ for which there exists a unique solution to the initial-value problem. Write down this solution.

(ii) State a range of γ for which there exist multiple solutions. Construct two different solutions.

2. (a) Suppose that $X(t)$ is a fundamental matrix of the linear differential system

$$x'(t) = A(t)\, x(t) \tag{1}$$

where $A(t)$ is an $n \times n$ matrix of period $T > 0$, i.e. $A(t+T) = A(t)$ for any t.

(i) Show that $X(t+T)$ is also a fundamental matrix, and is related to $X(t)$ via $X(t+T) = X(t)B$, where B is a constant matrix. Show that the determinant of B, det$B$, is given by

$$\det B = \exp\left\{\int_0^T \operatorname{tr} A(s)\, d\, s\right\}.$$

[You may use, without proof, the identity

$$\det\Big(X(t)\Big) = \det\Big(X(t_0)\Big)\exp\left\{\int_{t_0}^t \operatorname{tr} A(s)\, d\, s\right\} \quad \text{valid for any } t \text{ and } t_0.$$]

(ii) Define the Floquet multipliers and Floquet exponents.

(iii) Deduce that there exists at least one unbounded solution to (1) if

$$\int_0^T \operatorname{tr} A(s)\, d\, s > 0.$$

(b) Consider now the differential equation

$$u''(t) + \epsilon\Big(1 + \nu \sin 2t\Big)u'(t) + u(t) = 0,$$

where $\epsilon > 0$, ν are constants. Set $u = e^{\mu t} q(t)$, where μ is a constant and $q(t)$ is a periodic function, and then expand as follows for small values of ϵ:

$$\begin{aligned} q(t) &= q_0(t) + \epsilon q_1(t) + O(\epsilon^2), \\ \mu &= \qquad\quad \epsilon\mu_1 + O(\epsilon^2), \end{aligned}$$

(i) Calculate $q_0(t)$.

(ii) Derive the equation satisfied by $q_1(t)$, and determine a relation between μ_1 and ν so that $q_1(t)$ is periodic. Deduce that there is an unbounded solution if $|\nu| > 2$.

[To speed up your calculation, you may use the following identities:

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)];$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)].]$$

3. (a) Consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

where A is a 2×2 real constant matrix.
Show that if trace(A) $= 0$ then the form

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

is exact and can be solved analytically. Give the function $F(x, y)$ whose level curves are trajectories in the (x, y)-plane.

Hint: Recall that a differential form of the form $dF = P(u, v)du + Q(u, v)dv$ is exact when $\frac{\partial P}{\partial v} = \frac{\partial Q}{\partial u}$.

(b) Let now A be the 2×2 matrix

$$A_\alpha = \begin{pmatrix} 0 & 1+\alpha \\ -1 & \alpha \end{pmatrix},$$

where α is a real parameter.

(i) Show that the origin is always a critical point of the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_\alpha \begin{pmatrix} x \\ y \end{pmatrix}$.

(ii) Explain how the choice of α affects the trajectories in phase plane and give the values of α (if any) where the behaviour change completely. Noting that $\ell :$ det$(\mathrm{A}_\alpha) =$ trace$(\mathrm{A}_\alpha) + 1$ defines a line in the (tr, det)–plane, draw a bifurcation diagram in the (tr, det)–plane illustrating how the trajectories change character as α increases from $-\infty$ to ∞.

(iii) In the case $\alpha = 0$ use the result of Part (a) to determine the curves that give the trajectories in phase plane. Sketch also typical trajectories for $\alpha = -2, -1$.

4. (a) For the plane autonomous system $\dot{x} = f(x)$, where f and x are two-dimensional vectors, prove that if $\operatorname{div} \mathrm{f}$ is strictly of one sign in a bounded simply connected region R, then there is no periodic solution in R.

(b) Consider the non-linear system

$$\dot{x}_1 = f_1(x_1, x_2) \equiv x_1(1 - x_1^2 - x_2^2 - a\exp(x_1^2)) - x_2,$$
$$\dot{x}_2 = f_2(x_1, x_2) \equiv x_1 + x_2(1 - x_1^2 - x_2^2 - a\exp(x_1^2)),$$

where $a \geq 0$ is a constant.

(i) Use part (a) to show that if $a \geq 1$ there is no periodic solution.

(ii) Show that in terms of the polar coordinates (r, θ), the system can be rewritten as

$$\dot{r} = r(1 - r^2 - a\exp(r^2\cos^2\theta)), \quad \dot{\theta} = 1,$$

where $x_1 + ix_2 = r(t)\exp(i\theta(t))$.

(iii) For the case $a = 0$, find the only limit cycle and determine its stability properties.

(iv) For $0 < a < 1$, construct an appropriate annular region and use the Poincaré-Bendixson Theorem to prove that the system has at least one limit cycle. [You may use the result: $\exp(-r^2) \geq 1 - r^2$.]

5. (a) State the Hopf Bifurcation Theorem for a planar autonomous system depending on a real parameter μ. Explain the relevance of the *genericity condition.*

(b) The FitzHugh's system

$$\frac{dx}{dt} = y + x - \frac{1}{3}x^3 - \mu,$$
$$\frac{dy}{dt} = a - x - by,$$

is a relatively simple model of nerve impulses. Here x is an excitability variable, y is a recovery variable, μ is an input which we here assume constant, and a, b are positive parameters, with $b < 1$.

(i) Give the Jacobian of the system at a steady state (x^*, y^*) and show that the steady state is stable if

$$b - (1 - x^{*2}) > 0, \quad 1 - b(1 - x^{*2}) > 0.$$

(ii) Show that the steady state of the model is unstable if x^* falls in the range $-\gamma < x^* < \gamma$, where $\gamma = \sqrt{1-b}$.

(iii) Show that if $\mu = \mu_c$, where

$$\mu_c = \frac{a - \gamma}{b + \gamma - \gamma^3/3}$$

there is a steady state $P = (\gamma, \frac{a-\gamma}{b})$, and that the Jacobian there has a pair of purely imaginary eigenvalues.

(iv) Show that if μ increases from μ_c, the eigenvalues at the steady state move into the right half plane, destabilising the steady state. Conclude that at $\mu = \mu_c$ a Höpf bifurcation takes place.