Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3M5/M4M5

Advanced Ordinary Differential Equations

Date: Thursday, 25th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider the initial-value problem

$$x' = f(x,t), \quad x(0) = 0, \tag{1}$$

where x is an n-dimensional vector, and f is an n-dimensional vector function defined for $t \in I : |t| \leq \alpha$ and $x \in D : |x| \leq \beta$.

(a) Define the Lipschitz condition satisfied by $f(x,t)$, and state its relevance to the uniqueness of the solution to the initial-value problem.

Show that the two-dimensional vector function

$$f(x,t) = \left(\sin t + x_2^2\,,\, \mathrm{e}^{-t^2}\, x_1 x_2\right)^\top,$$

where $|x| \leq \beta$ and $t \in (-\infty, +\infty)$, satisfies a Lipschitz condition. Give a value for the Lipschitz constant L.

In what range of t is the solution to the initial-value problem expected to exist according to Cauchy-Peano theorem?

(b) Suppose that $f(x,t)$ is a scalar function defined as $f(x,t) = t|x|^\gamma$ with $|x| \leq \beta$ and $|t| \leq \alpha$. For what value of γ does there exist a unique solution to the initial-value problem?

For what values of γ do there exist more-than-one solutions? Construct two different solutions.

(c) Suppose that $x(t)$ is a solution to (1), and $y(t)$ satisfies

$$y' = f(y,t) + \mu g(y,t), \quad y(0) = 0,$$

where f satisfies a Lipschitz condition with a Lipschitz constant L, and $g(x,t)$ is a continuous function of x and t and $|g| \leq 1$ for all x and $|t| \leq \alpha$. Show that

$$|x(t) - y(t)| \;\leq\; |\mu| t \,+ L \int_0^t \Big|(x(s) - y(s)\Big|\, ds$$

for $0 \leq t \leq \alpha$. Hence show that

$$|x(t) - y(t)| \leq \frac{|\mu|}{L}(\mathrm{e}^{Lt} - 1)\,.$$

Comment on the implication of this result on the dependence of the solution on the parameter.

2. (a) For the linear differential equations

$$\begin{aligned} x_1'(t) &= (\nu + \cos t) x_1 \,, \\ x_2'(t) &= (-\nu + \cos t) x_2 + x_1 \,, \end{aligned}$$

where ν is a constant, find the fundamental matrix $X(t)$.

Hence obtain the matrix $B = X(0)^{-1} X(2\pi)$.

Calculate the characteristic multipliers and characteristic exponents.

For what value of ν is there a periodic solution?

(b) A pendulum with a periodically varying length is described by the equation

$$(1 + \epsilon \cos 2t) u''(t) - 2\epsilon \sin 2t \, u'(t) + \delta u(t) = 0 \,,$$

where ϵ and δ are constants.

Set $u = \mathrm{e}^{\mu t} q(t)$, where μ is a constant and $q(t)$ is a periodic function, and then expand as follows for small values of ϵ:

$$\begin{aligned} q &= q_0(t) + \epsilon q_1(t) + O(\epsilon^2) \ , \\ \mu &= \epsilon \mu_1 + O(\epsilon^2) \ , \\ \delta &= 1 + \epsilon \delta_1 + O(\epsilon^2) \ . \end{aligned}$$

Calculate $q_0(t)$.

Derive the equation satisfied by $q_1(t)$ and determine a relation between μ_1 and δ_1 so that q_1 is periodic.

Sketch the regions of instability in the (δ, ϵ)-plane near to $\delta = 1$ for small ϵ.

Explain the concept of subharmonic parametric resonance with reference to the solution that you have found.

[To speed up your calculation, you may use the following identities:

$$\begin{aligned} \cos\alpha \cos\beta &= \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] \,; \\ \sin\alpha \sin\beta &= -\frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)] \ ; \\ \sin\alpha \cos\beta &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \ . \ ] \end{aligned}$$

3. (a) Suppose that the initial-value problem

$$x' = f(x) \ , \quad x(t_0) = x_0 \ .$$

has a solution $x(t)$ for all $t \geq t_0$.

Define the Liapunov stability of $x(t)$.

Suppose that $x(t)$ is a periodic function of t, representing a closed orbit Γ. Define the orbital stability of Γ.

(b) Consider the nonlinear plane system

$$\left.\begin{aligned} x_1'(t) &= \ x_2 + \gamma x_1 + (2x_1^2 + x_2^2)x_1 x_2^2 \,, \\ x_2'(t) &= \ -2x_1 + \gamma x_2 - (2x_1^2 + x_2^2)x_2 + (2x_1^2 + x_2^2)x_2^3 \,, \end{aligned}\right\}$$

where γ is a constant.

(i) What conclusion may you draw about the nature of the steady solution $(0,\ 0)$ based on a linearised stability analysis?

(ii) For the case $\gamma = 0$, construct a Liapunov function of the form $V = ax_1^2 + x_2^2$ (where a is a constant that you are expected to determine) for (x_1, x_2) in a suitable neighbourhood of $(0, 0)$.

Show that the steady solution $(0,\ 0)$ is uniformly stable.

By using La Salle's Invariance Principle, show further that $(0, 0)$ is asymptotically stable, and hence determine the nature of $(0,\ 0)$.

(iii) For $\gamma = 0$, deduce that there exists a periodic orbit $2x_1^2 + x_2^2 = 1$, and that this periodic orbit is unstable.

(iv) Sketch the trajectories for $\gamma = 0$ in the phase plane.

4. (a) State the Poincaré-Bendixson Theorem for orbits in a phase plane.

(b) Consider the nonlinear system

$$\begin{aligned} x_1'(t) &= x_2 + x_1(1 - 2a - x_1^2 - x_2^2)\,, & (1) \\ x_2'(t) &= -x_1 + x_2(1 - x_1^2 - x_2^2)\,, & (2) \end{aligned}$$

where a is a constant.

Show that if $a > 1$ there is no periodic solution.

Show that, in terms of the polar coordinates (r, θ), the system (1) and (2) can be written as

$$r' = r(1 - r^2 - 2a\cos^2\theta)\ , \quad \theta' = -1 + a\sin 2\theta,$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$.

For the case $a = 0$, find the limit cycle, and determine its stability.

For $-1 < a < \frac{1}{2}$, by constructing an appropriate annular region and using the Poincaré-Bendixson Theorem, prove that the system has at least one periodic solution. [Hint: consider $0 \leq a < \frac{1}{2}$ and $-1 < a < 0$ separately.]

Calculate the period of the periodic solution(s).

5. (a) Consider a general plane system

$$\begin{cases} x' = & f(x, y, \mu), \\ y' = & g(x, y, \mu), \end{cases}$$

where f and g are sufficiently smooth functions of x, y and μ, with μ being a real parameter.

Define a critical point $\Big(x_0(\mu), y_0(\mu)\Big)$ of the system.

Define a Hopf bifurcation point, μ_0 say, of the parameter μ, explaining your definition explicitly in terms of the relevant partial derivatives f_x, f_y, g_x and g_y.

Explain the implication of the genericity condition $\gamma \neq 0$ for the stability of (x_0, y_0), where

$$\gamma \equiv \frac{d}{d\,\mu}(f_x + g_y) \text{ evaluated at } (x, y) = (x_0, y_0), \quad \mu = \mu_0 \,.$$

Suppose that (x_0, y_0) is stable for $\mu < \mu_0$, and $\gamma > 0$. Explain supercritical and subcritical Hopf bifurcations by means of suitable bifurcation diagrams.

(b) The so-called Brusselator is a model for certain chemical reactions, and it consists of equations

$$x' = a - (\mu + 1)x + x^2 y,$$
$$y' = \mu x - x^2 y,$$

where x and y are concentrations ($x, y \geq 0$), and a and μ are positive parameters.

Find the critical point (x_0, y_0) of the system.

Derive the condition that parameters a and μ have to satisfy for a Hopf bifurcation to occur.

For $a = 1$, sketch the bifurcation diagrams of y_0 against μ.

Suppose that the Hopf bifurcation is supercritical. Sketch the phase-plane diagrams before and after the bifurcation, indicating any periodic orbit.