1. Consider the initial-value problem

$$x' = f(x,t), \quad x(t_0) = x_0,$$

where x is an n-dimensional vector, and f an n-dimensional vector function defined for $t \in I : |t - t_0| \leq \alpha$ and $x \in D : |x - x_0| \leq \beta$.

(a) State Cauchy-Peano Theorem concerning the existence of a solution to the initial-value problem.

(b) Define Lipschitz condition satisfied by $f(x,t)$, and state its relevance to the uniqueness of the solution to the initial-value problem.

Show that the following vector functions (i) and (ii) satisfy a Lipschitz condition and give in each case a value for the Lipschitz constant L.

(i) $f(x,t) = \left(x_1 x_2,\, \mathrm{e}^{-x_1^2} + (\sin t) x_2^2\right)^T$, where $|x| < M$ and $t \in (-\infty, +\infty)$.

(ii) $f(x,t) = \left(x_2(1 + x_2^2)^{-1},\, \mathrm{e}^{-|x_2|} + x_1\right)^T$, where x in the whole plane.

[You may use without proof the inequality: $\mathrm{e}^{-r} \geq 1 - r$ for any $r \geq 0$.]

(c) Suppose that $y(t)$ satisfies

$$y' = f(y,t), \quad y(t_0) = y_0 \quad \text{with} \quad |y_0 - x_0| \leq \beta,$$

and f satisfies a Lipschitz condition with a Lipschitz constant L. Show that

$$|x(y) - y(t)| \leq |x_0 - y_0| + L \int_{t_0}^{t} \Big|(x(s) - y(s)\Big| ds.$$

for $t \geq t_0$. Hence show that

$$|x(t) - y(t)| \leq |x_0 - y_0| \exp\{L(t - t_0)\}.$$

Comment on the implication of this result on the dependence of the solution on the initial data.

2. (a) Consider the linear differential system

$$x'(t) = A(t)\,x(t) \tag{1}$$

where $A(t)$ is an $n \times n$ matrix of period $T > 0$, i.e. $A(t+T) = A(t)$ for any t.

Let $X(t)$ be a fundamental matrix of the system. State how $X(t+T)$ and $X(t)$ are related.

Define the characteristic (Floquet) multipliers and exponents.

Show that if ρ is a characteristic multiplier, there exists a solution $x(t)$ such that $x(t+T) = \rho x(t)$.

Explain how one could calculate numerically the characteristic exponents to determine the structure of general solution.

(b) Suppose that the matrix $A(t)$ in (1) takes the form

$$A(t) = a(t)I + C,$$

where $a(t)$ is a scalar periodic function, i.e. $a(t+T) = a(t)$, I is a unit matrix and C a constant matrix. Let $x(t) = y(t)\exp\{\int_0^t a(s)ds\}$. Deduce the equation satisfied by $y(t)$. Hence or otherwise show that there exists a periodic solution with period T, if C has an eigenvalue λ such that

$$\lambda + \frac{1}{T}\int_0^T a(t)dt = 0.$$

Give the condition for the subharmonic resonance.

3. (a) State the Linearised Stability Principle for a steady solution, x^c, of a nonlinear autonomous system,

$$x' = f(x);$$

explain the relevance of any matrix eigenvalue problem involved.

(b) The nonlinear plane system

$$\begin{aligned} x_1'(t) &= x_2, \\ x_2'(t) &= -x_1 - x_1^3 - a x_1^2 x_2, \end{aligned}$$

has a steady solution $(0, 0)$, where a is a constant. What conclusion may you draw about the nature of the steady solution $(0,\ 0)$ based on a linearised stability analysis?

Write this system into a second-order system. Hence or otherwise construct a Liapunov function $V(x_1, x_2)$ to show that the steady solution $(0,\ 0)$ is uniformly stable for $a \geq 0$.

By using La Salle's Invariance Principle, show further that for $a > 0$ the steady solution $(0, 0)$ is asymptotically stable, and hence determine the nature of $(0,\ 0)$.

(c) Determine the nature of $(0, 0)$ for $a = 0$. Sketch the trajectories for both $a = 0$ and $a > 0$ in the phase plane.

4. (a) For the plane autonomous system

$$x' = f(x)$$

where $x = (x_1,\, x_2)^T$ and $f = (f_1,\, f_2)^T$ are two-dimensional vectors, prove that if $\mathrm{div} f$ is strictly of one sign in a region R, then there is no periodic solution that lies entirely in R.

(b) A nonlinear oscillator is described by the equation

$$u'' + \epsilon[u^2 - a]u' + u + \epsilon u^3 = 0$$

where ϵ and a are constants.

Show that there is no periodic solution for $a < 0$ using the result in (a).

In order to use the Poincaré-Lindstedt method to find the periodic solution for $|\epsilon| \ll 1$ and $a > 0$, the variable $\tau = \omega t$ is introduced. Derive the equation satisfied by $u = u(\tau)$.

Expand $u(\tau)$ and ω as follows

$$\begin{aligned} u &= u_0(\tau) + \epsilon u_1(\tau) + \dots\ , \\ \omega &= 1 + \epsilon\omega_1 + \dots\ . \end{aligned}$$

Find the equation for u_0 and show that it has the solution $u_0 = A_0 \cos\tau$.

Derive the equation satisfied by u_1, and determine the values of ω_1 and A_0.

Determine the stability of this periodic solution by considering

$$\int_0^T \mathrm{div} f(u(t))dt$$

for a suitable function f and T which you need to specify.

5. (a) For the plane system

$$\begin{aligned} x' &= \mu x - y, \\ y' &= -y + x^3, \end{aligned}$$

obtain the critical points (x_0, y_0), determine their stability properties and find the value of μ at any bifurcation point including a Hopf bifurcation point.

Sketch the bifurcation diagram of x_0, the first component of the critical points (x_0, y_0), against μ; label the stable and unstable branches.

(b) Suppose that the system is modified to

$$\begin{aligned} x' &= \mu x - y, \\ y' &= -y + x^3 + a, \end{aligned}$$

by adding to the second equation a constant term, which represents a small imperfection.

Show that there is one critical point for $\mu < \mu_c = 3(a/2)^{2/3}$ and three critical points for $\mu > \mu_c$. Determine their stability properties and sketch the bifurcation diagram of x_0 against μ; label the stable and unstable branches.