UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2007

This paper is also taken for the relevant examination for the Associateship.

M3M4 / M4M4

TECHNIQUES OF COMPLEX VARIABLE THEORY

Date: Friday, 18th May 2007 Time: 10am-12pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Evaluate

$$I = \int_0^\infty \frac{ln x \quad dx}{(a + x^2)}$$

by first constructing a complex integral and then using contour integration.

Hence or otherwise evaluate

$$\int_0^\infty \frac{dx}{(a + x^2)^2} \quad .$$

2. $F_N(a)$ is defined by

$$F_N(a) = \int_0^\pi \frac{\cos(N\theta) d\theta}{1 - 2a\cos\theta + a^2}$$

where $a \geq 0$ and $a \neq 1$ with N a positive integer. Show that

$$F_N(a) = \tfrac{1}{2}\, Re\,(G_N\,(a))$$

where

$$G_N(a) = \int_0^{2\pi} \frac{e^{iN\theta} d\theta}{1 - 2a\cos\theta + a^2}$$

Evaluate $G_N(a)$ by contour integration, distinguishing the cases $0 \leq a < 1$ and $a > 1$. Hence find $F_N(a)$.

Why does the analysis fail when $a = 1$?

3. The function $u(r,t)$ satisfies the differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} \qquad 0 \le r \le 1$$

the initial conditions

$$u(r,0) = 0 \quad \frac{\partial u(r,0)}{\partial t} = 1, \qquad 0 \le r \le 1$$

and the boundary conditions

$$u(1,t) = 1 \qquad t > 0$$

and that $u(0,t)$ is finite.

Show that the Laplace transform of u can be expressed in the form

$$\overline{u}(r,s) = r^{-1}\overline{w}(r,s)$$

where

$$\frac{d^2\overline{w}}{dr^2} - s^2\overline{w} = 0 \quad .$$

Hence determine $\overline{u}(r,s)$,

show that

$$u(r,t) = A_0 + \frac{2}{r}\sum_{N=1}^{\infty} B_N \sin(Nr\pi)\cos(N\pi t)$$

and determine the coefficients A_0 and B_N .

4. Use the Fourier transform over x to reduce the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \alpha^2 U = 0$$

to an ordinary differential equation. Solve this equation subject to the boundary conditions

$$\begin{aligned} U(x,0) &= 0 \\ U(x,1) &= \delta(x) \end{aligned}$$

where $\delta(x)$ is the Dirac delta function.

Show that the transform $\overline{u}(\zeta, y)$ has no branch points and evaluate $U(x,y)$ by contour integration to get an expression of the form

$$U(x,y) = \sum_{N=1}^{\infty} (-1)^{N+1} A_N \exp\left(-x\sqrt{\alpha^2 + N^2\pi^2}\right) \sin(Ny)$$

and determine A_N .

5. Obtain an asymptotic expansion of the integral

$$I = \int_0^{\infty} e^{-tx/2}\, x^t\, e^{-tx^3/6}\, dx$$

as $t \to \infty$ in the form

$$I = \frac{e^{-\alpha t}}{\sqrt{t}} \left(A_1 + 0\left(\frac{1}{t}\right)\right)$$

and determine A_1 and α .