Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3M4/M4M4

Techniques of Complex Variable Theory

Date: Thursday, 1st June 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Use contour integration to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{(\pi^2 - 4x^2)} \, dx \, .$$

2. Use the Laplace transform

$$\overline{u}(x,s) = \int_0^{\infty} u(x,t) e^{-st} dt \, ,$$

to show that the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin(\sigma t) \, ,$$

subject to the initial conditions

$$u(x,0) = 0 \, , \quad \frac{\partial u}{\partial t}(x,0) = 0 \, ,$$

can be written in the form

$$\overline{u}(x,s) = F_1(s) + A \sinh(sx) + B \sinh(s(L-x)) \, .$$

Determine $F_1(s)$, A and B such that the boundary conditions

$$u(0,t) = 0 = u(L,t)$$

are satisfied.

Identify the poles of the function $\overline{u}(x,s)$ and indicate how one might invert this transform. Evaluate only those residues at poles $s = \pm i\sigma$, and hence calculate this part of the solution in the form

$$u(x,t) = \sin(\sigma t) \; G(x) \, .$$

Determine $G(x)$ and verify that this part of the solution satisfies the above differential equation. Why is this not a complete solution?

3. (i) The Fourier transform $F(w)$ of a function $f(x)$ is defined as

$$F(w) = \int_{-\infty}^{\infty} e^{iwx} f(x) dx \,,$$

and the inverse as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwx} F(w) dw \,.$$

Prove that if

$$h(x) = \int_{-\infty}^{\infty} f(y) g(x-y) dy$$

then

$$H(w) = F(w) G(w) \,,$$

all transforms being assumed to exist and $H(w)$, $G(w)$ being the Fourier transforms of of $h(x)$ and $g(x)$ respectively.

(ii) Use a Fourier transform over x to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the region $y > 0$ with the boundary condition

$$u(x, y) = f(x) \quad \text{when } y = 0 \,.$$

Assuming that $f(x) \to 0$ as $x \to \pm\infty$, determine the Fourier transform $U(w, y)$ of $u(x, y)$; $F(w)$ being the Fourier transform of the boundary condition $f(x)$.

(iii) If $G(w) = e^{-|w|y}$, determine its inverse Fourier transform $g(x, y)$. Use the inverse of the convolution theorem (i), or otherwise, to show that

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(t)\, 2y}{(x-t)^2 + y^2} \, dt$$

is the solution of problem (ii).

4. By using the substitution $t = \dfrac{s}{\sqrt{x}}$, reduce the integral

$$I = \int_0^\infty \exp(-xt - \frac{1}{t})\,(1+t)dt$$

to the form

$$I = \int_0^\infty \exp\{-\sqrt{x}\phi(s)\}\,f(s)ds$$

and determine an expansion for I for $x \gg 1$ of the form

$$I = \frac{e^{-2\sqrt{x}}}{x^{\frac{3}{4}}}\left(A + \frac{B}{\sqrt{x}} + O(\frac{1}{x})\right)$$

and find the constants A and B.

5. State (without proof) a set of sufficient conditions for Watson's lemma to hold; that is

$$\int_0^B e^{-\lambda x} x^\mu f(x)dx \sim \frac{a_0\Gamma(\mu+1)}{\lambda^{\mu+1}} + \ldots + \frac{a_M\Gamma(\mu+M+1)}{\lambda^{\mu+M+1}}$$

as $\lambda \to \infty$, where B is a positive constant. Also determine the coefficients $a_0, a_1, \ldots, a_M$.

Use Watson's lemma to show that

$$\int_0^{\frac{\pi}{2}} e^{x\cos\beta}\cos(2\beta)d\beta \sim e^x\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(x^{-\frac{1}{2}} - \frac{a}{8}\,x^{-\frac{3}{2}} + \frac{b}{128}\,x^{-\frac{5}{2}} + \ldots\right)$$

as $x \to \infty$, and determine a and b.

[You may assume that

$$(1)\quad \Gamma(\lambda) = \int_0^\infty e^{-p}p^{\lambda-1}dp\,; \qquad (2)\quad \Gamma(\lambda+1) = \lambda\Gamma(\lambda)\,; \qquad (3)\quad \Gamma(\frac{1}{2}) = \sqrt{\pi}\,.$$]