Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M3M4/M4M4 Techniques of Complex Variable Theory

Date: Tuesday 31st May 2005 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Using a "keyhole" contour show that

$$\int_0^\infty \frac{x^P dx}{(x+a)(x+b)} = \frac{\pi}{\sin(P\pi)} \frac{1}{b-a} f(a,b) .$$

where $0 < a < b$ and $-1 < P < 0$. Determine the function $f(a,b)$.

Hence or otherwise show that

$$\int_0^\infty \frac{x^P dx}{(x+2)(x+1)^2} = \frac{\pi}{\sin(P\pi)} (1 + P - 2^P) .$$

2. The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} , \qquad 0 \leq x \leq 1 ,$$

with the initial conditions

$$u(x,0) = 0 , \qquad \frac{\partial u}{\partial t}(x,0) = 1 , \qquad 0 \leq x \leq 1 ,$$

the boundary conditions

$$u(1,t) = 0 , \qquad t > 0 ,$$

and the condition that $u(0,t)$ is finite.

Show that the Laplace transform of u can be expressed in the form

$$\widehat{u}(x,s) = x^{-1}\widehat{v}(x,s) ,$$

where

$$\frac{d^2\widehat{v}}{dx^2} - s^2\,\widehat{v} = -x , \qquad 0 \leq x \leq 1 .$$

Hence determine $\widehat{u}(x,s)$ and show that

$$u(x,t) = \frac{2}{\pi^2 x} \sum_{N=1}^{\infty} A_N \sin(N\pi x)\sin(N\pi t) .$$

Determine A_N .

3. (i) Evaluate the Fourier transform of

$$f_a(x) = \frac{1}{2a}\ \exp(-|x|/a)\,.$$

Hence show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\zeta x)\,d\zeta \;=\; \delta(x)\,, \qquad \text{(A)}$$

if $\delta(x)$ is considered as the limit as $a \to 0$ of $f_a(x)$.

(ii) Use the result (A) and properties of the δ function to show that the Fourier transform of $J_0(\lambda x)$, where

$$J_0(\lambda x) \;=\; \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\beta)\,d\beta\,,$$

and λ is real and positive and given by

$$\int_{-\infty}^{\infty}\exp(i\zeta x)\,J_0(\lambda x)\,dx \;=\; F(\zeta,\lambda)\,,$$

where ζ is real and

$$F(\zeta,\lambda) = 0, \qquad |\zeta| > \lambda\,.$$

Find $F(\zeta,\lambda)$ when $|\zeta| < \lambda$.

Hence show that the Fourier transform of $\cos(\mu x)J_0(\lambda x)$ is

$$\frac{1}{2}\left[F(\zeta+\mu,\lambda) + F(\zeta-\mu,\lambda)\right]$$

where μ is real and positive.

4. $\phi_1(x,y)$ satisfies $\dfrac{\partial^2\phi_1}{\partial x^2} + \dfrac{\partial^2\phi_1}{\partial y^2} = 0$ in $-\infty < x < \infty, \quad 0 \le y \le 1,$ and

$\phi_2(x)$ satisfies $\dfrac{\partial^2\phi_2}{\partial x^2} + \dfrac{\partial^2\phi_2}{\partial y^2} = 0$ in $-\infty < x < \infty, \quad 1 \le y \le 2,$ where

(1) $\phi_1 = 0$ on $y = 0$ for all x,

(2) $\dfrac{\partial\phi_2}{\partial y} = e^{-|x|}$ on $y = 2$ for all x, and

(3) $\phi_1 = \phi_2$ and $\mu_1\dfrac{\partial\phi_1}{\partial y} = \mu_2\dfrac{\partial\phi_2}{\partial y}$ on $y = 1$ for all x, where μ_1 and μ_2 are real constants.

Show that the Fourier transform

$$\Phi_1(\zeta, y) = \int_{-\infty}^{\infty} \exp(i\zeta x)\,\phi_1(x,y)\,dx$$

can be written as

$$\Phi_1(\zeta, y) = A_1\sinh(\zeta y)$$

and the Fourier transform $\Phi_2(\zeta, y)$ of $\phi_2(x,y)$ as

$$\Phi_2(\zeta, y) = A_2\sinh(\zeta(1-y)) + B_2\cosh(\zeta(1-y))$$

where condition (1) has been satisfied and A_1, A_2 and B_2 remain to be determined using conditions (2) and (3).

Hence determine $\Phi_1(\zeta, y)$ and show that this function has only poles with zero real parts in the complex ζ plane.

In the special case when $\sqrt{\dfrac{\mu_1}{\mu_2}} \ll 1$, show that $\phi_1(x,y)$ has the form

$$\phi_1(x,y) \;=\; \exp(-x)\,f_1(y) \;+\; \exp\left(-\sqrt{\mu_1/\mu_2}\;x\right)f_2(y)$$

as $x \to +\infty$, and determine $f_1(y)$ and $f_2(y)$.

5. Use the saddle point approximation or otherwise to get an expansion of the integrals

$$\begin{aligned} I_1 &= \int_0^\infty e^{-tx} x^t e^{-tx^2} dx \\ I_2 &= \int_0^\infty e^{tx} x^t e^{-tx^2} dx \end{aligned}$$

each in the form

$$I = \frac{be^{-at}}{\sqrt{t}} \left(1 + O\left(\frac{1}{\sqrt{t}}\right)\right)$$

as t tends to infinity, and determine the constants a and b in each case. The term $O\left(\frac{1}{\sqrt{t}}\right)$ denotes an error of order $\frac{1}{\sqrt{t}}$.

Hence or otherwise determine the leading term in the expansion of

$$\int_{-\infty}^{\infty} e^{-tx} |x|^t e^{-tx^2} dx$$

as t tends to infinity.