Imperial College London

UNIVERSITY OF LONDON

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| Course: | M3M4 |
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

# M3M4 Techniques of Complex Variable Theory

Date: Thursday, 20th May 2004 Time: $2$ pm –$4$ pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. By choosing a suitable contour for the integral

$$J = \oint_c \frac{lN(z)dz}{(1+z^4)}$$

evaluate the integral

$$I = \int_0^\infty \frac{dr}{(1+r^4)}.$$

2. (i) Given

$$F(t) = \int_0^t g(t-x)f(x)dx$$

Deduce the theorem

$$\overline{F}(s) = \overline{g}(s)\overline{f}(s)$$

where

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t)dt \quad \text{etc}$$

(ii) Use the above theorem to solve the integral equation

$$y(x) = e^{-x} + 3\int_0^x e^{2t-2x} y(t)dt$$

(iii) Solve the integral equation in (ii) by reducing it to a differential equation.

3. Use the Laplace transform to solve the equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2}$$

Subject to the conditions

$$1)\ \theta(0,t) = 0 \quad 2)\ \frac{\partial\theta}{\partial x}(L,t) = 0$$

$$3)\ \theta(x,0) = \alpha x \quad 4)\ \frac{\partial\theta}{\partial t}(x,0) = 0$$

Express your solution in the form

$$\partial(x,t) = \sum_{N=0}^{\infty} f_N(x)\cos\left(\frac{(2N+1)\pi t}{2L}\right)$$

and determine the functions $f_N(x)$.

4. The region $-\infty < x < \infty$, $-\infty < y < \infty$ is divided into two halves $y > 0$ and $y < 0$. For $y > 0$ there is a potential $\phi_1$ and an associated flux in the $y$ direction $\mu_1 \frac{d\phi_1}{\partial y}$ where $\mu_1$ is a constant.

For $y < 0$ the corresponding potential and flux are $\phi_2$ and $\mu_2 \frac{\partial \phi_2}{\partial y}$.

$\phi_1$ and $\phi_2$ satisfy the Laplace equation

$$\nabla^2 \phi_i = 0 \quad i = 1, 2.$$

The following conditions are to hold

(i)

$$\mu_1 \frac{\partial \phi_1}{\partial y} = \mu_2 \frac{\partial \phi_2}{\partial y} \quad \mathrm{on} y = 0$$

(ii)

$$\phi_1 - \phi_2 = \delta(x) \quad \mathrm{on} y = 0, \quad \delta(x) \mathrm{is the Dirac delta function.}$$

(iii)

$$\phi_1 \mathrm{tends to zero as} (x^2 + y^2) \to \infty \mathrm{in} y > 0$$
$$\phi_2 \mathrm{tends to zero as} (x^2 + y^2) \to \infty \mathrm{in} y < 0$$

Use the fourier transform over $x$ to find the functions $\phi_1$ and $\phi_2$.

5. The integral $I(\lambda)$ is defined for $\lambda$ real and positive, by

$$I(\lambda) = \int_{-\infty}^{\infty} \frac{e^{i\lambda z^2}}{(1+z^2} \, dz$$

Show that $z = 0$ is a saddle point and that the path of steepest descent is $x = y$ where $z = x + iy$ ($x$ and $y$ are real).

Show that

$$\int_{-R}^{R} \frac{e^{i\lambda z^2} dz}{(1+z^2)} = \int_{C_R} \frac{e^{i\lambda z^2} dz}{(1+z^2)}$$

Where $C_R$ consists of circular arcs joining $-R$ to $-Re^{\pi i/4}$ and $Re^{\pi i/4}$ to $R$ together with the straight line joining $-Re^{\pi i/4}$ to $Re^{\pi i/4}$ and draw the corresponding contour.

Hence show that if $\lambda >> 1$

$$I(\lambda) \sim e^{\pi i/4} \left(\frac{\pi}{\lambda}\right)^{1/2} \left(a - \frac{b}{\lambda} + ..\right)$$

and find $a$ and $b$.