1. (a) Define carefully the *integrals* of a PDE of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u),$$

and explain their importance.

(b) Find a first order quasilinear PDE whose general solution is given implicitly by

$$u^2 + xu + y = f(xy).$$

Find the solution of this PDE which satisfies

$$u = 0 \qquad \text{on} \qquad x = 1.$$

Discuss whether there is a solution for this PDE satisfying

$$u = 0 \qquad \text{on} \qquad x = 0.$$

If there is no solution in this case, explain the difference between the two problems.

2. (a) Explain what is meant by the *envelope* of the 1-parameter family of curves in the (x, y) plane:

$$y = f(x, s).$$

(b) Find two independent integrals of the PDE:

$$xyu_x + u^2u_y = -uy.$$

Find the solution of this PDE which satisfies $u = x$ on $y = 0$. Identify where this solution breaks down.

(c) Find the projection onto the (x, y) plane of the characteristic through the point $(t, 0, t)$. Find the envelope of these projected characteristics.

3. (a) Define and explain the terms elliptic, parabolic and hyperbolic for the second order quasilinear PDE

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} = 0.$$

(b) Show that the PDE:

$$u_{xx} + 2\frac{1}{F'(y)}u_{xy} + \frac{1}{F'(y)^2}u_{yy} = 0$$

is parabolic. Find and solve the ode for its characteristic coordinate. Here $F'(y)$ is differentiable, and its indefinite integral $F(y)$ is monotonic.

(c) Find also a suitable non-characteristic coordinate, and hence write the PDE in its canonical form.

4\. (a) Show that the initial value problem

$$u_{xy} = 0, \qquad y > 0,$$

$$u = f(x), \qquad y = 0,$$

$$u_y = g(x), \qquad y = 0,$$

is ill-posed.

(b) Show that the Dirichlet problem

$$\nabla^2 u = 0, \qquad x^2 + y^2 > 1,$$

with the boundary conditions

$$u = f(\theta), \qquad x = \cos(\theta), \quad y = \sin(\theta)$$

and

$$u \to 0 \qquad \text{as} \quad (x^2 + y^2) \to \infty,$$

has a unique solution. Write down a formula for the solution in terms of the Green's function for the problem, stating the defining properties of the Green's function $G(x, y; x', y')$.

(c) Show that in (b), if $f(\theta) = f(\pi - \theta)$, then the solution $u(x, y)$ is even in x. Hence solve the mixed boundary value problem:

$$\nabla^2 u = 0, \qquad x^2 + y^2 > 1, \quad x > 0,$$

with the boundary conditions

$$u = f(\theta), \qquad x = \cos(\theta), \quad y = \sin(\theta), \quad \theta \in [-\pi/2, \pi/2],$$

and

$$\frac{\partial u}{\partial x} = 0, \qquad x = 0,$$

and

$$u \to 0 \qquad \text{as} \quad (x^2 + y^2) \to \infty.$$

Again you are not required to write down the Green's function $G(x, y; x', y')$ explicitly.

5. (a) Show directly that $U(x,t;x',t')$, given by

$$U(x,t;x',t') = \frac{1}{2\sqrt{\pi(t-t')}} \exp\left(-\frac{(x-x')^2}{4(t-t')}\right), \qquad t > t',$$

$$U(x,t;x',t') = 0 \qquad t < t',$$

satisfies

$$U_t = U_{xx} + \delta(x-x')\delta(t-t').$$

What PDE does $U(x,t;x',t')$, considered as a function of (x',t'), satisfy?

(b) Construct a function $V(x,t;x',t')$ satisfying

$$V_t = V_{xx} + \delta(x-x')\delta(t-t'), \qquad x > 0,$$

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = 0.$$

(c) Solve the initial value problem

$$u_t = u_{xx}, \qquad x > 0, \quad t > 0,$$

$$u(x,0) = u_0(x),$$

$$u_x(0,t) = 0.$$