Imperial College
London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

# M3M3

## Partial Differential Equations

Date: Friday, 12th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Define the term *integral* for the first-order quasilinear pde

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u).$$

Explain how a first-order quasilinear pde may be constructed which possesses any 2 given independent integrals $I(x, y, u)$ and $J(x, y, u)$.

Construct a pde whose general solution is

$$f(xy + u, x + yu) = 0.$$

Give the equations of the general characteristic of this pde.

Find the equation of the projection of this characteristic onto the $(x, y)$ plane.

Find the solution of this pde satisfying

$$u = y \text{ on } x = 0.$$

2. Define the term *characteristic* in relation to the first-order quasilinear pde,

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u).$$

Find two independent integrals for the pde

$$x(y + u)u_x - y(x + u)u_y = (y - x)u.$$

Hence find the general solution of this equation and find the particular solution satisfying

$$u = 1 \text{ on } y = 1.$$

Find the curve $C$ on which this solution becomes complex.

Find the family of characteristics through the points $(x_0, 1, 1)$, and explain how they are related to $C$.

3. (i) Define the term *characteristic* for the system of two coupled linear first-order pde:

$$A\mathbf{u}_x + B\mathbf{u}_y = \mathbf{0},$$

where $A$ and $B$ are $2 \times 2$ matrices and $\mathbf{u}$ is a $2$-component vector. Hence explain the meaning of the terms 'elliptic', 'parabolic' and 'hyperbolic' in the context of this system. Further, explain the same terms for the second order pde

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = 0.$$

For this second-order system, Cauchy data is given as follows:

$$\mathbf{u} = \mathbf{v}(x),$$
$$\text{and} \quad \mathbf{u}_y = \mathbf{w}(x) \quad \text{on} \quad y = Y(x).$$

State whether, and under what conditions, there is a uniquely defined Taylor series for $u(x,y)$ in the neighbourhood of the curve.

If these conditions are met, so that the Taylor series for $u(x,y)$ is uniquely defined, does this mean that the boundary value problem is well-posed? If not, give an example of an ill-posed problem satisfying these conditions.

(ii) State in which regions the pde

$$u_{xx} = x^3 u_{yy}$$

is elliptic, parabolic or hyperbolic.

In the hyperbolic region, reduce it to canonical form.

4. Write down the free-space Green's function $G_0(\mathbf{r},\mathbf{r}')$ for Laplace's equation in $\mathbb{R}^3$, with vanishing boundary conditions at infinity, as well as the pde which it satisfies.

Show how the use of Green's identity and the Green's function $G_0$ leads to a solution of Poisson's equation

$$\nabla^2 u = \rho(\mathbf{r})$$

with the boundary condition $u \to 0$ as $|\mathbf{r}| \to \infty$.

Explain how this method is adapted to solve a Dirichlet problem in a finite domain $\Omega$.

Write down the pde which must be satisfied by the new Green's function $G_D$, and explain how the Dirichlet boundary conditions for $u$ lead to a choice of the appropriate boundary conditions for $G_D$.

Show how Green's identity leads to a solution of Laplace's equation in $\Omega$, satisfying Dirichlet boundary conditions on $\partial\Omega$. In particular take $\Omega$ to be the interior of the sphere $|\mathbf{r}| < a$, with boundary conditions on $|\mathbf{r}| = a$:

$$\begin{aligned} u &= 1, \quad z > 0, \\ u &= 0, \quad z > 0, \end{aligned}$$

and show that

$$u(\mathbf{r}') = \int_{\theta=0}^{\pi/2} \left[ \frac{1}{2} \frac{a^2 - r'^2}{(a^2 + r'^2 - 2ar'\cos(\Theta))^{3/2}} \right] a \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, .$$

*Hint: You should note that if $F(\mathbf{r})$ is harmonic in the exterior of the sphere, $\mathbf{r} > a$, and $\lim_{r\to\infty} F(\mathbf{r}) = 0$, then if we define*

$$\widetilde{\mathbf{r}} = \frac{\mathbf{r}a^2}{|\mathbf{r}|^2},$$

*it follows that*

$$\widetilde{F}(\mathbf{r}) = \frac{a}{|\mathbf{r}|} F(\widetilde{\mathbf{r}})$$

*is harmonic in the interior of the same sphere, $\mathbf{r} < a$.*

5. (i) Show that the initial value problem for the heat equation

$$u_t = u_{xx}, \quad t > 0, \qquad u(x,0) = u_0(x), \qquad u(x,t) \to 0, \quad |x| \to \infty$$

has a unique solution.

Discuss whether your proof can be adapted to the region $t < 0$. If it cannot, explain where the argument breaks down.

(ii) Show directly that the function

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp(-\frac{(x-x')^2}{4t}) u_0(x')\, \mathrm{d}x'$$

is a solution to this initial value problem.

(iii) Explain how you would adapt this solution to the initial-boundary value problem:

$$\begin{aligned} u_t &= u_{xx}, \quad x > 0, \quad t > 0, \\ u(x,0) &= u_0(x), \quad x > 0, \\ u(0,t) &= 0, \\ u(x,t) &\to 0, \quad x \to \infty\,. \end{aligned}$$

Write down the solution to this problem explicitly.