1. Define carefully the *characteristics* of a pde of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u),$$

and explain their importance.

Find the general solution of the pde:

$$(3x^2y - 2xu)u_x + (yu - 3xy^2)u_y = xyu,$$

carefully explaining the steps in your method.

Hence find the solution of this pde which satisfies

$$u = 1 \qquad \text{on} \qquad y = 1.$$

2. Explain what is meant by the *envelope* of the 1-parameter family of curves in the (x, y) plane:

$$y = f(x, s).$$

A pde has integrals

$$I = uy - x^2,$$

$$J = ux.$$

Find its characteristics passing through each point of the line Γ, parametrised as follows:

$$(x, y, u) = (s, 1, -1).$$

Find the projections of these characteristics by eliminating u, and then find the envelope of the projected characteristics.

Find the solution of the pde, with boundary data given on the line Γ, and verify directly that it is singular on this envelope.

3. Define the terms elliptic, parabolic and hyperbolic for a second order quasilinear pde.

State whether the pde

$$u_{xx} - \frac{1}{F'(y)^2} u_{yy} = 0$$

is elliptic, parabolic or hyperbolic, and reduce it to canonical form. Here $F(y)$ is a twice differentiable strictly monotonic function and $F'(y)$ is its derivative.

Write down the pde and its canonical form explicitly in the case that $F(y) = \exp(y)$.

4. (a) Show that the initial value problem

$$u_{xx} + u_{yy} = 0, \qquad y > 0,$$
$$u = f(x), \qquad y = 0,$$
$$u_y = g(x), \qquad y = 0,$$

is ill-posed.

(b) Show that the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \qquad y > 0,$$
$$u = f(x), \qquad y = 0,$$
$$u \to 0, \qquad x^2 + y^2 \to 0$$

has a unique solution.

(c) Solve the Dirichlet problem (b) in the case

$$f(x) = x \qquad |x| < 1,$$
$$f(x) = 0 \qquad |x| > 1.$$

5. (a) The heat equation

$$u_t = u_{xx}$$

is to be solved in the domain $t > 0$, with initial condition

$$u(x, 0) = f(x).$$

Show that the solution can be written as

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} \exp(-(x-x')^2/4t) f(x') \mathrm{d}x'$$

State clearly any results you use.

(b) Show that solutions of the heat equation can be mapped into non-trivial solutions of Burger's equation

$$v_t + v v_x = v_{xx}$$

by the Cole-Hopf transformation

$$v = \alpha u_x / u,$$

if α is chosen appropriately. You should find this value of α.

(c) By taking the singular initial data for the heat equation,

$$f(x) = \delta^{(n)}(x),$$

construct a solution $u(x,t)$ of the heat equation. Also construct its image under the Cole-Hopf transformation $v(x,t)$, which solves Burger's equation.

Hint: These solutions can be written more elegantly using the Hermite polynomials $H_n(x)$ *defined by*

$$H_n(x) = \exp(x^2)(-\frac{\mathrm{d}}{\mathrm{d}x})^n \exp(-x^2).$$