Imperial College London

UNIVERSITY OF LONDON

Course:	M3M3
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M3M3 Partial Differential Equation

Date: May 2004 Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. Find two integrals of the partial differential equation

$$x\,u_x - 2y\,u_y = x\,u,$$

and hence write down the general solution.

Find the solution which satisfies the boundary condition

$$u = \cos(y) \qquad \text{on } x = 1.$$

Consider also the boundary condition

$$u = \cos(y) \qquad \text{on } x = 0,$$

and explain carefully any differences between the two problems.

2. Define carefully the terms "integral" and "characteristic" for a 1st order quasilinear pde,

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

If I and J are differentiable functions of x, y and u, then write down the most general integral of the pde:

$$\begin{vmatrix} I_x & I_y & I_u \\ J_x & J_y & J_u \\ u_x & u_y & -1 \end{vmatrix} = 0.$$

A characteristic of this pde passes through a point $(x_0,\, y_0,\, u_0)$. What are the equations of this characteristic?

Find a pde with the integrals $(x^2 + y^2 + u^2)$ and xyu, and find the projected characteristic passing through $(1, y_0, 1)$.

Find the envelope of the projected characteristics.

3. A second order quasilinear pde is

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)\,u_{yy} = K(x,y,u,u_x,u_y)$$

Define the characteristics of this pde and explain what is meant by saying it is elliptic, parabolic or hyperbolic.

Reduce to canonical form the pde

$$(x^2-1)^2 u_{xx} = u_{yy}$$

in the region $x > 1$.

4. **i.** Prove the uniqueness theorem for the Dirichlet problem for Laplace's equation in a simply-connected plane region Ω:

$$\begin{cases} \nabla^2 u = 0 \quad , & \mathbf{x} \in \Omega \\ u = f(\mathbf{x}) \quad , & \mathbf{x} \in \partial\Omega \end{cases}$$

ii. The Dirichlet Green's function for Laplaces equation in the disc of radius a is polar coordinates is

$$G_0(r,\theta,r_0,\theta_0) = \frac{1}{4\pi}\ln\left[\frac{r^2 + r_0^2 - 2r\,r_0\cos(\theta-\theta_0)}{\frac{r^2 r_0^2}{a^2} + a^2 - 2r\,r_0\cos(\theta-\theta_0)}\right]$$

Find the Dirchlet Green's function for Laplace's equation in the semicircle $0 < r < a$, $0 < \theta < \pi$.

Hence solve the Dirichlet problem

$$\begin{aligned} \nabla^2 u &= 0 && 0 < r < a, 0 < \theta < \pi \\ u &= 1 && \text{on } \theta = 0; \\ u &= 0 && \text{on } \theta = \pi; \\ u &= 0 && \text{on } r = a. \end{aligned}$$

Give the solution in integral form.

5. The equation

$$u_t = u_{xx} \tag{1}$$

is to be solved in the quarter-plane

$$x > 0$$

$$t > 0$$

with boundary conditions

$$u(x,0) = u_0(x) \tag{2}$$

$$u(0,t) = w(t) \tag{3}$$

Prove that the solution is unique.

Discuss the analogous problem in the region $x > 0$, $t < 0$.

Show how the fundamental solution of the heat equation

$$G_0(x,t;x_0,t_0) = \begin{cases} \frac{\exp\left((x-x_0)^2 \,/\, 4(t-t_0)\right)}{2\sqrt{\pi(t-t_0)}}, & t > t_0 \\ 0, & t < t_0. \end{cases}$$

may be used to solve the initial and boundary value problem in $x > 0$, $t > 0$, (1), (2), (3).

Hence or otherwise solve

$$u_t = u_{xx} \qquad x > 0\ ,\ t > 0$$

with

$$u(x,0) = 0$$

$$u(0,t) = \frac{1}{\sqrt{t}}$$

Do not attempt to evaluate any solution in integral form.