**1.** Let $F_{ab}$ by the skew symmmetric tensor that satisfies the equations

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \tag{1}$$

$$\nabla_a F^{ab} = J^b. \tag{2}$$

If the tensor $F_{ab}$ is expressed through the covariant vector $A_a$ as

$$F_{ab} = \partial_a A_b - \partial_b A_a, \tag{3}$$

show that (3) is equivalent to

$$F_{ab} = \nabla_a A_b - \nabla_b A_a.$$

Hence show that (1) is identically satisfied and the covariant vector $A_a$ satisfies,

$$g^{bc}\nabla_b\nabla_c A^a - R^a_b A^b = J^a,$$

where $R^a_b$ is the Ricci tensor and $A^a$ is subjected to the condition,

$$\nabla_a A^a = 0.$$

$$\text{Hint :} \quad (\nabla_b\nabla_c - \nabla_c\nabla_b)V^a = R^a{}_{dbc}V^d.$$

***Turn over...***

**2.** From the Bianchi identities:

$$\nabla_a R^d{}_{ebc} + \nabla_b R^d{}_{eca} + \nabla_c R^d{}_{eab} = 0,$$

prove the contracted Bianchi identities:

$$\nabla_a G^{ab} = 0,$$

where $G^{ab} = R^{ab} - Rg^{ab}/2$ is the Einstein's tensor.

The Einstein's equation in non-empty space is

$$G^{ab} = kT^{ab}, \quad k = \text{constant}$$

where $T^{ab}$ is the energy-momentum tensor.

A cloud of non-interacting dust particles with density $\varrho_0(x^0, x^1, x^2, x^3)$ has energy-momentum tensor

$$T^{ab} = \varrho_0 v^a v^b,$$

where

$$v^a = \frac{dx^a}{ds}$$

is the tangent vector to the path $x^a = x^a(s)$. From the contracted Bianchi identities show that $x^a = x^a(s)$ satisfies the geodesic equations,

$$\frac{d^2 x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0.$$

**3.** The Schwarzchild solution has line element

$$(ds)^2 = \left(1 - \frac{2m}{r}\right) c^2 (dt)^2 - \frac{(dr)^2}{1 - \frac{2m}{r}} - r^2 \left[(d\theta)^2 + \sin^2\theta (d\phi)^2\right],$$

where $(r, \theta, \phi)$ are the standard polar coordinates. A massive particle falls radially towards the surface $r = 2m$ from $R(> 2m)$. Show that the time it takes to reach $r = 2m + \varepsilon$ from $R$ as measured by a stationary observer is

$$ct \sim 2m \ln\left(\frac{2m}{\varepsilon}\right),$$

for small $\varepsilon$. Show that it takes only a finite time from the point of view of an observer attached to the massive particle. How long will it take for a photon to fall radially towards $r = 2m$?

**4.** The line element or the first fundamental form of a two dimensional surface with coordinates $(x, y)$ is given by

$$(ds)^2 = [f(x) + g(y)]\left[(dx)^2 + (dy)^2\right],$$

where $f(x)$ is a function of $x$ only and $g(y)$ is a function of $y$ only. Determine the geodesic equations and show that the geodesics are level curves of the function $\Phi = \Phi(x, y)$ where

$$\frac{\partial \Phi}{\partial x} = \frac{1}{\sqrt{f(x) + \alpha}},$$

$$\frac{\partial \Phi}{\partial y} = -\frac{1}{\sqrt{g(y) - \alpha}}$$

and $\alpha$ is an arbitrary constant.

**5.** Find the time-like geodesic $x = f(t)$ in a two-dimensional Minkowski space-time with the line element

$$(ds)^2 = c^2(dt)^2 - (dx)^2.$$

Let $\tau(P, Q, x(t))$ be the proper time interval along any time-like curve $x(t)$ between the points $P$ and $Q$. If the geodesic $x = f(t)$ is perturbed into the curve $x = f(t) + \xi(t)$, with the same end points $P_1 = (ct_1, x_1)$ and $P_2 = (ct_2, x_1)$, $t_2 > t_1, s$ show that

$$\delta\tau(P_1, P_2) := \tau(P_1, P_2, f(t) + \xi(t)) - \tau(P_1, P_2, f(t))$$

$$= -\frac{1}{2}\int_{t_1}^{t_2} \frac{(\dot{\xi}/c)^2 dt}{\left[1 - (\dot{f} + \alpha\dot{\xi})^2/c^2\right]^{3/2}}, \quad 0 < \alpha < 1,$$

where $\dot{g} = \frac{dg}{dt}$. Use this result to discuss the twin paradox.

Hint: Taylor's Theorem with remainder states

$$f(t + h) = f(t) + h\dot{f}(t + \alpha h), \qquad 0 < \alpha < 1.$$

 