1. Let X^a and Y_b be contravariant and covariant vectors respectively. Show that $T^a_b := X^a Y_b$ is a type (1,1) rank two tensor. From the covariant derivative of covariant and contravariant vectors show that

$$\nabla_c T^a_b = \partial_c T^a_b + \Gamma^a_{dc} T^d_b - \Gamma^d_{bc} T^a_d.$$

Furthermore show that if $S^a_{\ bc} := T^a_b V_c$, where V_c is a covariant vector, then

$$\nabla_d S^a_{\ bc} = \partial_d S^a_{\ bc} - \Gamma^e_{bd} S^a_{\ ec} - \Gamma^e_{dc} S^a_{\ be}.$$

Turn over...

2. The Einstein's equation in non-empty space is

$$G^{ab} = AT^{ab},$$

where $G^{ab} = R^{ab} - Rg^{ab}/2$ is the Einstein's tensor and T^{ab} is the energy-momentum tensor. State and prove the contracted Bianchi identities from the Bianchi identities:

$$\nabla_a R^d{}_{ebc} + \nabla_b R^d{}_{eca} + \nabla_c R^d{}_{eab} = 0.$$

A cloud of non-interacting dust particles with density $\varrho_0(x^0, x^1, x^2, x^3)$ has energy-momentum tensor

$$T^{ab} = \varrho_0 v^a v^b,$$

where

$$v^a = \frac{dx^a}{ds}$$

is the tangent vector to the path $x^a = x^a(s)$. From the contracted Bianchi identities show that $x^a = x^a(s)$ satisfies the geodesic equations,

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0.$$

3. The Schwarzchild solution has line element

$$(ds)^2 = \left(1 - \frac{2m}{r}\right) c^2 (dt)^2 - \frac{(dr)^2}{1 - \frac{2m}{r}} - r^2 \left[(d\theta)^2 + \sin^2\theta (d\phi)^2\right],$$

where (r, θ, ϕ) are the standard polar coordinates. A massive particle falls radially towards the surafce $r = 2m$ from $R(> 2m)$. Show that the time it takes to reach $r = 2m + \varepsilon$ from R as measured by a stationary observer is

$$ct \sim 2m \ln\left(\frac{2m}{\varepsilon}\right),$$

for small ε. Show it takes only a finite time from the point of view of an observer attached to the massive partticle. How long will it take for a photon to fall radially towards $r = 2m$?

4. The line element of a two dimensional surface with coordinates (x, y) is

$$(ds)^2 = [f(x) + g(y)]\left[(dx)^2 + (dy)^2\right],$$

where $f(x)$ is a function of x only and $g(y)$ is a function of y only. Determine the geodesic equations and show that the geodesics are level curves of $\psi(x, y)$ where

$$\frac{\partial \psi}{\partial x} = \frac{1}{\sqrt{f(x) + \alpha}},$$

$$\frac{\partial \psi}{\partial y} = -\frac{1}{\sqrt{g(y) - \alpha}}$$

and α is an arbitrary constant.

5. A vector λ^a, $a = 0, 1, 2, 3$, orthogonal to the tangent vector $dx^a/d\tau$, where $d\tau$ is the propertime interval, is being parallel transported around a circle of fixed radius r in the Schwarzchild space-time where the line element reads;

$$(ds)^2 = \left(1 - \frac{2m}{r}\right) c^2 (dt)^2 - r^2 (d\varphi)^2.$$

Show that the tangent vector is

$$\left(\frac{dx^a}{d\tau}\right) = (A, 0, 0, A\Omega),$$

where

$$A = \sqrt{\frac{r}{r - 3m}} \quad \text{and} \quad \Omega = c\sqrt{\frac{m}{r^3}}.$$

From the parallel transport equations of λ^a deduce that

$$\frac{d\lambda^1}{d\tau} - \frac{r\Omega}{A}\lambda^3 = 0,$$

$$\frac{d\lambda^2}{d\tau} = 0,$$

$$\frac{d\lambda^3}{d\tau} + \frac{A\Omega}{r}\lambda^1 = 0.$$

Show that after one revolution in which the coordinate time has elasped by $\Delta t = 2\pi/\Omega$, the final spatial direction of λ^a deviates from the intial direction by

$$2\pi \left(1 - \sqrt{1 - \frac{3m}{r}}\right).$$

You may choose τ and t so that when τ is zero t is also zero and assume that initially λ^a is radial.

The non-zero connections are

$$\Gamma^0_{10} = \frac{m}{r}\frac{1}{1 - 2m/r}, \quad \Gamma^1_{00} = \frac{mc^2}{r^2}\left(1 - \frac{2m}{r}\right), \quad \Gamma^1_{33} = -r\left(1 - \frac{2m}{r}\right), \quad \Gamma^3_{13} = \frac{1}{r}.$$

The orthogonality condition is $g_{ab}\frac{dx^a}{d\tau}\lambda^b = 0$.