1. (i) Show that the average energy of a system with partition function Z is given by

$$E = -\frac{\partial \ln(Z)}{\partial \beta}.$$

Consider a gas of identical and indistinguishable mono-atomic non-interacting particles of mass m.

(ii) Calculate the partition function for this ideal gas.

(iii) Calculate the heat capacity of the ideal gas.

(iv) Use the thermodynamics identity

$$dU = TdS - pdV$$

to derive the ideal gas equation of state.

2. Consider a time signal $f(t)$ that can assume two values A and $-A$. The signal switches from one value to the other with probability ν per time.

(i) Show that the probability that the signal has not switched during the a time interval of duration t is given by

$$p_0(t) = e^{-\nu t}.$$

(ii) Show, e.g. by induction, that the probability that $f(t)$ has changed value exactly n times during the time t is given by

$$p_n(t) = \frac{(\nu t)^n}{n!} e^{-\nu n t}.$$

(iii) Calculate the auto-correlation function

$$C(T) = \langle f(t) f(t+T) \rangle_t$$

for all $t \in \mathbb{R}$.

(iv) Calculate the power spectrum.

3. Consider percolation on a d-dimensional hyper-cubic lattice. Let P_∞ denote the probability that an occupied site belongs to the infinite cluster.

(i) Use mean field theory to determine the critical percolation density p_c and the order parameter exponent β in

$$P_\infty \propto (p - p_c)^\beta.$$

(ii) Let D denote the fractal dimension of the spanning cluster. Derive the scaling relation.

$$D = d - \beta/\nu,$$

where ν is the correlation length exponent.

(iii) Show that

$$P_\infty = 1 - \frac{1}{p}\sum_{s=1}^{\infty} s n_s.$$

(iv) For one dimensional percolation find n_s and use the expression in part (iii) to show that $P_\infty = 0$ for all $p < 1$.

4. Consider the Ising model given by the Hamiltonian

$$H = -\sum_{i=1}^{N} S_i h,$$

where $S_i \in \{-1, 1\}$ and h is an external magnetic field.

(i) Calculate the partition function.

(ii) Calculate the average energy and derive the limiting behaviour for $T \to 0$ and $T \to \infty$.

(iii) Calculate the heat capacity $C(T)$ and show that $C(T)$ is maximal for $T = h/(kx_0)$, where x_0 is the solution to $x \tanh(x) = 1$.

(iv) Show that the magnetic susceptibility $\chi = \partial M/\partial h$ is proportional to $1/T$ for $h \ll kT$. [You may use without proof: $\tanh(x) = x - x^3/3 + \cdots$ for small x.]

5. Consider the random neighbour version of a sandpile model on a d-dimensional hypercube in the limit of large system size $N \to \infty$.

(i) Explain how the dynamics can be mapped on to a branching process.

Assume that adding a grain to a site makes the site relax with probability ν.

(ii) Determine the branching probabilities p_k that a relaxing site generates k new relaxations.

The generator function $g_{Z_n}(s)$ for the stochastic variable Z_n ($=$ number of sites excited in time step n) satisfies the relation

$$g_{Z_n}(s) = g_{Z_1}(g_{Z_{n-1}}(s)).$$

(iii) Use this relation to express the average $\langle Z_n \rangle$ in terms of the average branching ratio.

(iv) Show that the random neighbour sandpile is critical when $\nu = 1/(2d)$.