Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M3A5 Statistical Mechanics and Complex Systems

Date: Tuesday 31st May 2005 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The energy of a system consisting of N particles is given by

$$H = \sum_{i=1}^{N} \epsilon_i \ ,$$

where the single particle energies ϵ_i can assume only the two values $\epsilon_i = 0$ or $\epsilon_i = E > 0$. Assume the system to be in contact with a heat bath at temperature T.

(i) Calculate, as function of T, the average number of particles with energy E.

(ii) Calculate Helmholtz's free energy.

(iii) Calcualte the entropy.

(iv) Calculate the heat capacity.

2. Consider N non-interacting classical 3 dimensional harmonic oscillators. The energy of each oscillator is given by

$$E = \frac{\mathbf{p}^2}{2m} + \frac{\alpha}{2}\mathbf{r}^2. \tag{1}$$

(i) Show that $\langle Aq^2 \rangle = \dfrac{kT}{2}$ where Aq^2 is either equal to $p_i^2/2m$ or equal to $\alpha_i r_i^2/2$ for $i = 1,\ 2,\ 3.$

(ii) Determine the heat capacity.

When the harmonic oscillator is treated quantum mechanically its energy in Equation (1) is replaced by

$$E_{nmk} = \hbar\,\omega\,(\frac{3}{2} + n + m + k),$$

where $n,\ m,\ k \in \{0, 1, 2, 3, \ldots\}$.

(iii) Derive the partition function for N non-interacting quantum oscillators.

(iv) Show that the classical result for the heat capacity is recovered in the limit $T \to \infty$.

3. (i) Explain how the self-consistent mean field equation

$$\langle S\rangle = \tanh[\beta(B\mu + \frac{Jq}{2}\langle S\rangle)]$$

is obtained from the Ising energy functional

$$E = -\frac{J}{2}\sum_{i=1}^{N}\sum_{j=1}^{q} S_i S_{n_i(j)} - \mu B\sum_{i=1}^{N} S_i \ .$$

(ii) Use the self-consistent equation to determine the mean field transition temperature T_c .

(iii) Derive the temperature dependence of the order parameter $\langle S\rangle$ in the vicinity of T_c. (*You may use without proof:* $\tanh x = x - \frac{x^3}{3} + \cdots$ *for small* x.)

(iv) Consider the one dimensional Ising chain and explain briefly Landau's argument for the absence of a phase transition at any non-zero temperature in this system. Why does this imply that mean field fails qualitatively for the one dimensional Ising chain?

4. Consider percolation in one dimension. Let p denote the probability that a site is occupied. The probability that a randomly chosen site belongs to a cluster of size s is equal to $sn_s(p)$, where $n_s(p)$ is the cluster size distribution.

(i) Determine $n_s(p)$.

(ii) Calculate, as function of p, the average cluster size $\langle s\rangle$.

Consider the correlation function defined by

$g(|\,\mathbf{r}-\mathbf{r}_0|) \ = \ \gamma$ prob(site $\mathbf{r}$ is occupied and belongs to the same cluster as $\mathbf{r}_0$),

where γ is a constant.

(iii) Determine how the correlation function depends on $r = |\,\mathbf{r}-\mathbf{r}_0|$.

(iv) Determine the correlation length as function of p and show that the correlation length exponent is $\nu = 1$.

5. Let $n(x,t)$ denote the density of particles diffusing along the x-axis. The time evolution of $n(x,t)$ is controlled by the equation

$$\frac{\partial n}{\partial t} \;=\; \gamma \frac{\partial^2 n}{\partial x^2} + g(x,t).$$

(i) Express the density $n(x,t)$ in terms of the Fourier transform $\widehat{g}(k,\omega)$ given by

$$\widehat{g}(k,\omega) \;=\; \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt \; g(x,t)\, e^{-i(kx+\omega t)} \; .$$

Consider the fluctuations $N(t)$ in the density at a given position $x_0 > 0$ defined by

$$N(t) = n(x_0,t) - \langle n(x_0,t)\rangle_t \; .$$

(ii) For $\omega \neq 0$ express the Fourier transform $\widehat{N}(\omega)$ in terms of $\widehat{g}(k,\omega)$.

Assume that the source term is confined to act at the origin, i.e. of the form $g(x,t) = \delta(x)\,\chi(t)$.

(iii) Show that

$$|\widehat{N}(\omega)|^2 \;=\; \frac{|\widehat{\chi}(\omega)|^2}{4\gamma\omega} \exp\{-\sqrt{\frac{2\omega}{\gamma}} x_0\} \; .$$

(iv) Assume that the temporal signal $\chi(t)$ consists of white noise and determine the frequency scale below which the power spectrum of $N(t)$ behaves like $1/f$ noise.