1. A particle is in a quantum state with the wave–function of the form:

$$\psi(x) = \frac{1}{(2\pi)^{1/4}\sqrt{a}} \exp\left(\frac{i}{\hbar} p_0 x - \frac{x^2}{4a^2}\right) ,$$

where p_0 and a are parameters with dimensions of momentum and length, respectively.

(i) Write down the probability distribution for the co-ordinate x. Hence calculate $< x >$, $< x^2 >$ and the standard deviation $\Delta x = \sqrt{< (x- < x >)^2 >}$.

(ii) Find the eigenstates of the momentum operator $\widehat{p} = -i\hbar d/dx$ and determine the probability distribution of the momentum eigenvalues p. Hence obtain $< \widehat{p} >$, $< \widehat{p}^2 >$ and Δp.

(iii) What is the Heisenberg uncertainty relation in the state $\psi(x)$?

[The Gaussian integral $\int\limits_{-\infty}^{\infty} dx e^{-\alpha x^2} = \sqrt{\pi/\alpha}$, where α is a positive constant, can be used without proof.]

2. A particle of mass m moves in the potential of the form

$$U(x) = \lambda\delta(x) + \begin{cases} 0, & \text{for} \quad x < 0 \\ U_0, & \text{for} \quad x \geq 0 \end{cases}$$

with $U_0 > 0$.

Plot the potential. Using the continuity conditions for the wave-function at $x = 0$, find the reflection coefficient $R(E)$ as a function of energy for $E > U_0$. Obtain the limiting form of $R(E)$ as $E \to \infty$. Determine $R(U_0)$ and argue what $R(E)$ should be equal to for $0 < E < U_0$. Hence plot $R(E)$ for all $E > 0$.

3. A particle of mass m moves in the potential of the form

$$U(x) = \begin{cases} -F_1 x, & \text{for} \quad x < 0 \\ F_2 x, & \text{for} \quad x \geq 0 \end{cases}$$

with $F_{1,2} > 0$.

Plot the potential. Determine the classical turning points $x = a_{1,2}$. Using the Bohr-Sommerfeld quantisation rule,

$$\int\limits_{a_1}^{a_2} dx p(x) = \pi\hbar\left(n + \frac{1}{2}\right) ,$$

where n is a non-negative integer and $p(x)$ is the classical momentum, find the energy eigenvalues E_n in the quasi-classical approximation. Find the limiting form of the density of states $\Delta E_n = E_{n+1} - E_n$ for large values of n. Is it an increasing or decreasing function of n? Explain why.

4. The Hamiltonian of a two–dimensional harmonic oscillator of mass m and frequency ω is given by:

$$\widehat{H}_0 = \sum_{i=1}^{2} \left[\frac{\widehat{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \widehat{x}_i^2 \right] ,$$

where $\widehat{x}_i$ and $\widehat{p}_i$ are the canonical coordinate and momentum operators with the commutation relation $[\widehat{x}_i, \widehat{p}_j] = i\delta_{ij}$ (in units such that $\hbar = 1$).

The ladder operators are defined by:

$$\widehat{a}_i = \frac{1}{\sqrt{2m\omega}}(m\omega\widehat{x}_i + i\widehat{p}_i), \quad \widehat{a}_i^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega\widehat{x}_i - i\widehat{p}_i) \ .$$

Using the above commutation relation find out what commutation relations $[\widehat{a}_i, \widehat{a}_j^\dagger]$ are satisfied by the ladder operators. Derive the Hamiltonian of the two–dimensional harmonic oscillator in terms of the ladder operators.

Using the commutation relations, calculate the ground-state averages $\langle 0|\widehat{p}_1\widehat{p}_2^3|0\rangle$ and $\langle 0|\widehat{p}_1^2\widehat{p}_2^2|0\rangle$. [Hint: operators $\widehat{a}_i$ annihilate the ground state $\widehat{a}_i|0\rangle = 0$.]

5. A particle has the angular momentum $l = 1$ (in units of $\hbar$). Let $|m\rangle$ be the basis of states, which diagonalises the z-projection of the angular momentum operator: $\widehat{l}_z|m\rangle = m|m\rangle$, $m = 1, 0, -1$.

Find the eigenvalues of the x-projection, $\widehat{l}_x$, of the angular momentum operator. Also obtain the corresponding eigenfunctions in terms of linear combinations of $|m\rangle$'s. Hence determine the probability distribution for all possible values of $\widehat{l}_x$ in the state $|m\rangle$.

Hint: the $\widehat{l}_x$ matrix in the $|m\rangle$ basis is given by

$$\widehat{l}_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} .$$