Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3A4

Quantum Mechanics I

Date: Thursday, 1st June 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. A particle of mass m is subject to a potential $V(x)$ in one dimension given by

$$V(x) = \begin{cases} \infty & x < 0 \\ -V_0 & 0 \leq x \leq a \\ 0 & x > a, \end{cases}$$

where $V_0 > 0$.

Show the condition for a stationary bound state of energy E is that it satisfies the equation,

$$-\cot(ka) = \frac{1}{k}\sqrt{\frac{2mV_0}{\hbar^2} - k^2},$$

where $k^2 = \dfrac{2m}{\hbar^2}(E + V_0)$.

Find the minimum value of V_0 for a bound state to exist.

2. The operator $\widehat{a}$ and its Hermitian conjugate $\widehat{a}^\dagger$ have a commutator, $[\widehat{a}, \widehat{a}^\dagger] = 1$, and the operator $\widehat{N} = \widehat{a}^\dagger\widehat{a}$ has an eigenstate $|\alpha\rangle$ with eigenvalue α, $\widehat{N}|\alpha\rangle = \alpha|\alpha\rangle$.

Show that

(i) the state $\widehat{a}^\dagger|\alpha\rangle$ is an eigenstate of $\widehat{N}$ with an eigenvalue $(\alpha + 1)$,

(ii) the state $\widehat{a}|\alpha\rangle$ is an eigenstate of $\widehat{N}$ with an eigenvalue $(\alpha - 1)$,

(iii) the expectation value of $\widehat{N}$, $\langle\widehat{N}\rangle$, is such that $\langle\widehat{N}\rangle \geq 0$.

Deduce that α must be a positive integer n or zero.

If $|n\rangle$ are normalised eigenstates of $\widehat{N}$, show that

$$\widehat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \text{and} \quad \widehat{a}|n\rangle = \sqrt{n}|n-1\rangle.$$

Use the eigenstates of $\widehat{N}$ to deduce a matrix representation of the operator $\widehat{x} = \widehat{a} + \widehat{a}^\dagger$.

3. (i) If Hermitian operators $\widehat{A}$ and $\widehat{B}$, which correspond to physical observables A and B, have a commutator $[\widehat{A}, \widehat{B}] = i\widehat{C}$, prove the uncertainty relation,

$$\langle(\widehat{A} - \langle\widehat{A}\rangle)^2\rangle\langle(\widehat{B} - \langle\widehat{B}\rangle)^2\rangle \geq \frac{\langle\widehat{C}\rangle^2}{4} \,.$$

(ii) The state $|l, m\rangle$ is an eigenstate of $\widehat{\mathbf{l}}^2$, and $\widehat{l}_z$, such that

$$\widehat{\mathbf{l}}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle, \qquad \widehat{l}_z|l,m\rangle = m\hbar|l,m\rangle,$$

where $\widehat{\mathbf{l}} = (\widehat{l}_x, \widehat{l}_y, \widehat{l}_z)$ is the angular momentum operator, l is a positive integer or zero, and $m = -l, -l+1, \ldots, l-1, l$. Show that in the state $|l, l\rangle$ the expectation values $\langle\widehat{l}_x\rangle = \langle\widehat{l}_y\rangle = 0$, and $\langle\widehat{l}_x^2 + \widehat{l}_y^2\rangle = \hbar^2 l$, and prove that this value for $\langle\widehat{l}_x^2 + \widehat{l}_y^2\rangle$ is the minimum possible consistent with the uncertainty relation.

[You may assume the angular momentum commutation relations, $[\widehat{l}_x, \widehat{l}_y] = i\hbar\widehat{l}_z$, $[\widehat{l}_z, \widehat{l}_x] = i\hbar\widehat{l}_y$, and $[\widehat{l}_y, \widehat{l}_z] = i\hbar\widehat{l}_x$.]

4. (i) A system is described by a Hamiltonian H, and is such that the corresponding operator $\widehat{H}$ has a complete orthonormal set of discrete eigenstates, $\{|n\rangle\}$ with corresponding eigenvalues $\{E_n\}$, such that $E_{n+1} \geq E_n$, where n is a positive integer or zero. If $\langle\psi|\widehat{H}|\psi\rangle$ is the expectation value with respect to a possible normalised wave-function ψ, show that the ground state energy E_0 satisfies the inequality,

$$\langle\psi|\widehat{H}|\psi\rangle \geq E_0 \,.$$

(ii) A particle of mass m moves in a one dimensional potential $V(x) = \lambda x^4$. By taking a trial wave-function of the form, $\psi = (\alpha/\pi)^{\frac{1}{4}} e^{-\alpha x^2/2}$, show that the ground state energy E_0 must be such that

$$E_0 \leq \frac{3\hbar^2}{8m}\left(\frac{6m\lambda}{\hbar^2}\right)^{1/3} \,.$$

[You may assume the result $\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$.]

5. The spherical harmonics, $Y_{l,m}(\theta_1, \phi_1)$ and $Y_{l',m'}(\theta_2, \phi_2)$, are normalised eigenstates of the total and z-component of angular momentum, $\widehat{\mathbf{l}}_i^2$ and $\widehat{l}_{i,z}$, for particles $i = 1$ and $i = 2$, respectively. For $l = l' = 1$, construct eigenstates $|L, M\rangle$ of the total angular momentum operators $\widehat{\mathbf{L}}^2$ and $\widehat{L}_z$, corresponding to the quantum numbers, $L = 2$ and $L = 1$, with $M = 1$ for both cases, where $\widehat{L} = \widehat{l}_1 + \widehat{l}_2$.

Calculate the expectation values of $l_{1,z}^2 + l_{2,z}^2$ in each of these states.

[You may assume the relation, $\widehat{l}_{\pm} Y_{l,m} = \hbar\sqrt{l(l+1) - m(m \pm 1)}Y_{l,m\pm1}$, where $\widehat{l}_{\pm} = \widehat{l}_x \pm i\widehat{l}_y$.]