1. A particle of mass m moves in one dimension on the interval $(-\infty, \infty)$ under the influence of the potential

$$V(x) = -\frac{\hbar^2 Q}{2m}\left(\delta(x-a) + \delta(x+a)\right),$$

where Q is a positive constant and $\delta(x)$ is the Dirac delta function. Show that the Schroedinger equation can be written in this form:

$$\psi''(x) + Q\left(\delta(x-a) + \delta(x+a)\right)\psi(x) - \gamma^2\psi(x) = 0,$$

where γ is related to the bound state energy $E(<0)$ via

$$\gamma = \sqrt{\frac{-2mE}{\hbar}}.$$

Show that the bound state energies can be determined from

$$Qa - \xi = f(\xi),$$

where

$$f(\xi) = \frac{\xi}{\tanh\xi},$$

if the wave function is odd, and

$$f(\xi) = \xi\tanh\xi,$$

if the wave function is even. Here $\xi = \gamma a$. From a graphical consideration, show that if $0 < Qa < 1$, there is only one bound state and the corresponding wave function is even and that there are two bound states for $1 < Qa < \infty$. Determine the parity of the wave function of the higher energy bound state.

2. A spin$-\frac{1}{2}$ particle in a magnetic field $\mathbf{B} = B\mathbf{i}$ (B is a constant) has the Hamiltonian,

$$\hat{H} = \mu\sigma_x B,$$

where μ is the Bohr magneton and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the allowed energies and the corresponding eigenvectors by solving the time-independent Schrödinger equation,

$$\hat{H}\phi = E\phi,$$

where ϕ is a two-component column vector. From this show that the time-dependent Schrödinger equation

$$i\hbar\frac{d\psi(t)}{dt} = \hat{H}\psi(t),$$

has the general solution

$$\psi(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-i\mu Bt/\hbar} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{e}^{+i\mu Bt/\hbar},$$

where c_1 and c_2 are arbitrary constants. If at time $t = 0$, the particle is in the eigenstate $\hat{S}_y$ with eigenvalue $\frac{\hbar}{2}$, show that the probability for the particle to remain in the same spin state at time t is $\cos^2 \mu Bt/\hbar$. What is the probability of the particle to be in the spin state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ given the same initial condition?

3. A particle with mass m in one dimension is described by the wave function $\Psi(x,t), \quad -\infty < x < \infty$. Define the expectation value $\langle \hat{A} \rangle$ of the operator $\hat{A}$. Use the Time Dependent Schrödinger equation to show that

$$\frac{d\langle \hat{A} \rangle}{dt} = \frac{i}{\hbar}\langle\, [\hat{H}, \hat{A}]\, \rangle.$$

From this deduce that

$$\frac{d^2\langle \hat{A} \rangle}{dt^2} = -\frac{1}{\hbar^2}\langle\, [\hat{H}, [\hat{H}, \hat{A}]]\, \rangle.$$

Suppose the particle has Hamiltonian,

$$\hat{H} = \frac{\hat{p}}{2m} + V(x),$$

show that Newton's second law is satisfied on "average":

$$m\frac{d^2\langle x \rangle}{dt^2} = \left\langle -\frac{dV}{dx} \right\rangle.$$

For a charged particle under the influence of a constant electric field $\mathcal{E}$, $V(x) = -e\mathcal{E}x$. Show that,

$$< x >= \frac{e\mathcal{E}}{2m}t^2 + at + b,$$

where a and b are arbitrary constant.

4. Let $\hat{A}$ and $\hat{B}$ be Hermitean operators with commutator, $[\hat{A}, \hat{B}] = i\hat{C}$. Show that

$$\langle \hat{A}^2 \rangle \langle \hat{B}^2 \rangle \geq \frac{\langle \hat{C} \rangle^2}{4}.$$

The Hamiltonian of a one-dimensional harmonic oscillator of mass m is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2.$$

Show that

$$\langle \hat{H} \rangle \geq \frac{\langle \hat{p}^2 \rangle}{2m} + \frac{m\omega^2 \hbar^2}{8\langle \hat{p}^2 \rangle}$$

and hence

$$\langle \hat{H} \rangle \geq \frac{\hbar\omega}{2}.$$

5. Define the orbital angular momentum operator $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ for a particle three dimensions. Deduce the commutation relation

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z.$$

The normalized angular momentum eigenstates ψ_m satisfy the equations

$$\hat{\mathbf{L}}^2 \psi_m = l(l+1)\hbar^2 \psi_m, \quad \hat{L}_z \psi_m = m\hbar \psi_m.$$

By considering $\langle \psi_m | \hat{L}_+^2 \psi_m \rangle$ and $\langle \psi_m | \hat{L}_-^2 \psi_m \rangle$, where

$$\hat{L}_\pm := \hat{L}_x \pm i\hat{L}_y,$$

show that

$$\langle \psi_m | \hat{L}_x^2 \psi_m \rangle = \langle \psi_m | \hat{L}_y^2 \psi_m \rangle$$

and find the value in terms of l and m. Show also that the real part of $\langle \psi_m | \hat{L}_x \hat{L}_y \psi_m \rangle$ is zero, and deduce that

$$\langle \psi_m | \hat{L}_x \hat{L}_y \psi_m \rangle = i\frac{m\hbar^2}{2}.$$