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Course:	M3A23/M4A23/MSc?
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Date:	April 3, 2007

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

M3A23/M4A23/MSc?

Specimen Paper

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

M3A23/M4A23/MSc?

Specimen Paper

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) (a) Give the definition of a contraction $F : \mathbb{R}^n \to \mathbb{R}^n$, where we view $\mathbb{R}^n$ as a metric space with the usual Euclidean metric.

(b) Prove the following:

Theorem: *Under the iterates of a contraction $F : \mathbb{R}^n \to \mathbb{R}^n$, all points in $\mathbb{R}^n$ converge exponentially to a unique fixed point.*

(ii) (a) Show that the map

$$x_{n+1} = F(x_n) := \frac{1}{2}\left(x_n + \frac{y}{x_n}\right),$$

describes an iterative algorithm (Newton's method) to approximate $\sqrt{y}$ with $y \in \mathbb{R}^+$. [Note: you do not need to derive Newton's method, just apply it.]

(b) Show that the domain of attraction of the fixed point $x = \sqrt{y}$ for the map F contains the interval $[\sqrt{y/3}, \infty)$. (The *domain of attraction* of a fixed point is the set of all points that converge to this fixed point under (forward) iteration of the map.) What can you say about the rate of convergence to this fixed point?

2. (i) Let $F : S^1 \to S^1$ be an orientation preserving circle diffeomorphism.

(a) Let $G : \mathbb{R} \to \mathbb{R}$ be a lift of F. Give the definition of a lift of a circle map and prove a statement about its (non-)uniqueness. Show that $G(x+1) - G(x) = 1$.

Let the rotation number ρ_F of a circle map F with lift G be defined as $\rho_F(x) := (\rho_G(x) \bmod 1) \in [0, 1)$, where

$$\rho_G(x) := \lim_{n\to\infty} \frac{G^n(x) - x}{n}, \quad x \in \mathbb{R}.$$

(b) Suppose that the rotation number exists. Prove that for any orientation preserving circle diffeomorphism F the rotation number ρ_F is independent of the initial condition (x, in the above definition) and the choice of the lift G.

(ii) Let $\phi \in S^1 \cong \mathbb{R}/\mathbb{Z} \cong [0, 1)$, and consider the non-autonomous time-periodic differential equation on the circle

$$\frac{d\phi}{dt} = f(\phi, t), \quad f(\phi, t) = f(\phi, t+1),$$

where f is a smooth vectorfield (C^∞).

(a) Show that the time-1 return map $\Phi : S^1 \to S^1$ is a diffeomorphism and a Poincaré map. Argue why Φ is orientation preserving. [You may use properties of the flow of a vector field without proof, but all relevant properties should be carefully stated.]

(b) Suppose that the rotation number ρ_Φ of the Poincaré map Φ is equal to $3/4$. Describe in words what this implies for the dynamics of Φ. What does it imply for the underlying flow of the nonautonomous vector field f? [No proofs needed, but be precise in your discussion.]

3. Consider the quadratic map

$$f_\lambda : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \lambda x(2-x),$$

with $\lambda > 0$.

(i) For small values of $\lambda > 0$, $x = 0$ is an asymptotically stable fixed point. Determine the parameter value at which this fixed point loses stability. Determine the type of local bifurcation that occurs. Is it a *generic* (typical) local bifurcation of one-dimensional maps with one parameter? [Motivate your answer.]

(ii) Let Σ be the set of all (half-)infinite sequences of 0s and 1s, each such a sequence denoted $\omega := \{\omega_i\}_{i\in\mathbb{N}}$ with $\omega_i \in \{0,1\}$. Let σ denote the shift operator on sequences in Σ, acting as

$$\sigma(\omega_0, \omega_1, \omega_2, \ldots) = (\omega_1, \omega_2, \omega_3, \ldots)$$

.

(a) Let $\lambda = 1 + \frac{1}{2}\sqrt{5}$. Show that the set U of bounded orbits of f_λ is contained in the subset $[0,2]$ and is given by

$$U = \cap_{n\in\mathbb{N}} f_\lambda^{-n}([0,2]),$$

where $f^{-1}(x) := \{y \mid f(y) = x\}$.

(b) Show, when $\lambda = 1 + \frac{1}{2}\sqrt{5}$, that f_λ is uniformly expanding on $f_\lambda^{-1}([0,2])$, and that there is a homomorphism (one-to-one map) $h : \Sigma \to U$ such that $h \circ \sigma = f_\lambda \circ h$. Does it follow from this result that f_λ is topologically conjugate to σ? (Motivate your answer.)

(c) Use the result in (b) to show that, when $\lambda = 1 + \frac{1}{2}\sqrt{5}$, f_λ^n has exactly 2^n fixed points. How many points in Δ_0 never leave Δ_0 under iteration by f_λ?

4. Let Σ_3 denote the set of bi-infinite sequences $\{\omega_i\}_{i\in\mathbb{Z}}$ whose entries ω_i are taken from a set of three symbols, for instance $\{0, 1, 2\}$. Consider the cylinder

$$C_{\alpha_{1-n},\ldots,\alpha_{n-1}} := \{\omega \in \Sigma \mid \omega_i = \alpha_i,\ |i| < n\}.$$

Let

$$d(\omega, \omega') := \sum_{m\in\mathbb{Z}} \frac{\delta(\omega_m, \omega'_m)}{4^m},$$

where $\delta(a, b) = 0$ if $a = b$ and $\delta(a, b) = 1$ if $a \neq b$.

(i) (a) Show that d is a <u>metric</u> on Σ_3.

(b) Consider Σ_3 as a metric space with metric d. Show that the cylinder $C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$ is a ball in Σ around α and determine its radius.

(ii) Give for each of the following, an example of a topological Markov chain on Σ_3, by means of its transition matrix or Markov graph, that has this property:

1. a transitive topological Markov chain
2. a topological Markov chain that is not transitive
3. a topologically mixing topological Markov chain
4. a topological Markov chain that is not topologically mixing

(iii) Prove that every transitive topological Markov chain on Σ_3 is topologically mixing.

5. Let $F : S^1 \to S^1$ be an expanding circle map.

(i) Define what it means for F to be <u>expanding</u>. What does it imply for its <u>degree</u>?

(ii) Show that the number of fixed points of F^n is equal to $|d|^n - 1$, where d is the *degree* of F. (Hint: first prove the result when $n = 1$, and then prove that $\deg(F^n) = (\deg(F))^n$.)

(iii) Show that an expanding circle map F is <u>topologically mixing</u>.

(iii) Give a definition for F to have <u>sensitive dependence on initial conditions</u>.

(iv) Prove that topological mixing for F implies sensitive dependence on initial conditions.