Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3A23/M4A23

Introduction to Chaos Theory

Date: Monday, 8th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a map.

   (a) Define the (forward) orbit $\mathcal{O}(x)$ of a point x.

   (b) Define the omega-limit $\omega(x)$ of x.

   (c) Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with distinct complex conjugate eigenvalues $\alpha \pm i\beta$ with $\alpha^2 + \beta^2 = 1$.

      (i) Give a specific example of a map A for which $\omega(x)$ is a circle for every $x \neq 0$.

      (ii) Give a specific example of a map A for which $\omega(x)$ is *not* a circle for every x.

   (d) Give an example of a nonlinear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ for which $\omega(x) = \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit circle if and only if $x \in \mathbb{S}^1$.

2. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a circle homeomorphism.

   (a) Define the lift of f.

   (b) Define the rotation number of f.

   (c) Define what it means to say that two circle homemorphisms f and g are topologically conjugate.

   (d) State Denjoy's Theorem about circle maps.

   (e) Sketch the construction of Denjoy's counterexample and explain what it is a counterexample to.

3. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (2x + 3x^5y^3 - x^{27}y^5,\ 3y - 213x^{67}y + xy^{21})$$

and

$$g(x, y) = (27x + 13y - 29yx^{20},\ 27y - 13x + 2x^4y^{24} - xy^{89})$$

   (a) Calculate the derivative maps of f and g.

   (b) Calculate the derivatives $Df_{0,0}$ and $Dg_{0,0}$ of f and g at the fixed points at the origin.

   (c) Cite relevant results from the course to deduce that f and g are topologically conjugate on some neighbourhood of the origin.

4. Consider the map $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 3x & \text{if } 0 \le x \le 1/2\,, \\ 3(1-x) & \text{if } 1/2 \le x \le 1. \end{cases}$$

Let

$$\Lambda = \{x : f^k(x) \in [0,1] \text{ for all } k \ge 0\}.$$

Prove that there exists a bijection $h : \Lambda \to \Sigma_2^+$, where $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ is the set of one-sided sequences of the symbols 0 and 1.

5. Let

$$\Sigma_{10} = \{0, 1, 2, \ldots, 9\}^{\mathbb{Z}}$$

be the space of bi-infinite sequences of symbols $0, 1, 2, .., 9$ with the metric

$$d(\underline{x}, \underline{y}) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{10^{|i|}}\ .$$

(a) (i) Give the general dynamical definition of the stable and unstable sets $W^s(\underline{x})$ and $W^u(\underline{x})$ of an arbitrary point $\underline{x} \in \Sigma_{10}$.

(ii) Give an equivalent definition purely in terms of the properties of the symbol sequences.

(b) (i) Give the general metric definition of what it means for a set $X \subset \Sigma_{10}$ to be *dense* in Σ_{10}.

(ii) Give an equivalent definition purely in terms of the properties of the symbol sequences.

(c) Deduce the following properties.

(i) The stable (and unstable) sets of any point are dense in Σ_{10}.

(ii) The stable sets of two distinct points $\underline{x}$ and $\underline{y}$ either coincide or are disjoint.

(iii) The stable set $W^s(\underline{x})$ and the unstable set $W^u(\underline{y})$ of two distinct points always intersect and $W^s(\underline{x}) \cap W^u(\underline{y})$ is dense in Σ_{10}.

(d) Let $\underline{z} \in \Sigma_{10}$ be a sequence which never contains the digit 3. Show that the orbit of $\underline{z}$ is not dense in Σ_{10}.