Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M3A23/M4A23 Introduction to Chaos

Date: Wednesday 1st July 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a map.

(a) Define the (forward) orbit $\mathcal{O}(x)$ of a point x.

(b) Define the omega-limit $\omega(x)$ of x.

(c) Define a class of linear maps $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that the omega-limit of every point (except for the origin) is a circle.

(d) Define a linear map such that the omega-limit $\omega(x)$ of every point x is a (strictly non-circular) ellipse.

2. Let $f,\, g : \mathbb{S}^1 \to \mathbb{S}^1$ be two circle maps.

(a) Define what it means to say that f and g are topologically conjugate.

(b) Define the C^1 distance between f and g.

(c) Define what it means to say that f is C^1 structurally stable.

(d) Give an example of a circle map which is structurally stable. Give some justification for this fact in a few sentences, possibly citing results from the course.

(e) Give an example of a circle map which is not structurally stable. Again, give some justification for this fact in a few sentences, possibly citing results from the course.

3. Let $f,\, g : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = (2x + 3xy^2 - x^2y^3,\ 3y - 7x^2y + xy^5)$$

and

$$g(x,y) = (27x - 13y - 7yx^5,\ 27y + 13x + 2x^2y^3 - xy^6)$$

(a) Calculate the derivative maps of f and g.

(b) Calculate the derivatives $Df_{0,0}$ and $Dg_{0,0}$ of f and g at the fixed points at the origin.

(c) Cite relevant results from the course to deduce that f and g are topologically conjugate on some neighbourhood of the origin.

4. Consider the map $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/2 \\ 4(1-x) & \text{if } 1/2 \le x \le 1. \end{cases}$$

Let
$$\Lambda = \{x : f^k(x) \in [0,1] \text{ for all } k \ge 0\}.$$

Prove that there exists a bijection $h : \Lambda \to \Sigma_2^+$ where $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ is the set of one-sided sequences of the symbols 0 and 1.

5. Let
$$\Sigma_3 = \{0,\ 1,\ 2\}^{\mathbb{Z}}$$

be the space of bi-infinite sequences of symbols $0, 1, 2$ with the metric

$$d(\underline{x}, \underline{y}) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{3^{|i|}}.$$

(a) (i) Give the general dynamical definition of the stable and unstable sets $W^s(\underline{x})$ and $W^u(\underline{x})$ of an arbitrary point $\underline{x} \in \Sigma_3$.

(ii) Give an equivalent definition purely in terms of the properties of the symbol sequences.

(b) (i) Give the general metric definition of what it means for a set $X \subset \Sigma_3$ to be *dense* in Σ_3.

(ii) Give an equivalent definition purely in terms of the properties of the symbol sequences.

(c) Deduce the following properties:

(i) The stable (and unstable) sets of any point are dense in Σ_3;

(ii) The stable sets of two distinct points $\underline{x}$ and $\underline{y}$ either coincide or are disjoint;

(iii) The stable set $W^s(\underline{x})$ and the unstable set $W^u(\underline{y})$ of two distinct points always intersect and $W^s(\underline{x}) \cap W^u(\underline{y})$ is dense in Σ_3.

(d) Let $\underline{z} \in \Sigma_3$ be a sequence which never contains the digit 1. Show that the orbit of $\underline{z}$ is not dense in Σ_3.