Imperial College London

UNIVERSITY OF LONDON

Course:	M3A23
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M3A23 Introduction to Chaos Theory

Date: Thursday, 20th May 2004 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to I$ be a C^1 map. State what it means for a fixed point $p \in I$ to be attracting.

   Suppose that $p \in I$ is a fixed point for f and that $|f'(p)| < 1$. Give a direct proof of the fact that p is attracting.

2. Let $\sum = \sum_2^+ = \{0,1\}^{\mathbb{N}}$ denote the set of all one-sided infinite sequences of symbols 0 and 1 and let d be the metric on $\sum$ defined by

$$d(\underline{a}\,,\,\underline{b}) = 2^{-N}\text{where } N = \min\ \{k \in \mathbb{N} :\, a_k \equiv b_k\}\,.$$

   1. Define the shift map $\sigma : \sum \to \sum$.
   2. Define the adding machine map $A : \sum \to \sum$.
   3. Show that A is continuous.
   4. Show that the inverse A^{-1} is well defined by giving the explicit rule for A^{-1}, and show that A^{-1} is continuous.
   5. Give the definition of a dense orbit in $\sum$ and characterise the property of having a dense orbit in terms of the symbol sequence.
   6. Give examples of sequences in $\sum$ whose orbits are dense for A and for σ respectively. Justify your answer.
   7. Are σ and A topologically conjugate? Why?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the map given by $f(x) = x^2 - 3.75$. Let p_0 denote the positive fixed point for f and let $I = [-p_0 , p_0]$. Let

$$\Lambda = \{x : f^n(x) \in I \,\forall\, n \geq 0\}.$$

Prove that there exists a bijection $h : \Lambda \to \sum$ where $\sum$ is the set of one-sided infinite sequences of the symbols 0 and 1.

4. Let I be a closed interval and let $f , g : I$ be C^1 maps.

   1. Say what it means for f , g to be topologically conjugate.
   2. Say what it means for f , g to be C^1 ε close.
   3. Say what it means for f to be structurally stable.
   4, Show that the bijection h in the previous question is a topological conjugacy.
   5. Deduce that f is C^1 structurally stable.

5. Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear vector field defined by

$$v(x\,,\,y\,,\,z) = (v_1\,x\,,\,v_2\,y\,,\,v_3\,z)$$

for constants $v_1\,,\,v_2\,,\,v_3 \in \mathbb{R}$.

1. Calculate the position of a point $x_0\,,\,y_0\,,\,z_0$ after time t, under the flow induced by the vector field v.

2. Let $\mathbb{D}^3 = [-1\,,\,1]^3$ be a cube containing the origin. Suppose that

$$v_1 > 0,\, v_2 < 0,\, v_3 < 0.$$

Calculate the time it takes for a point on the top of the cube $\{z = 1\}$ to reach one of the sides $\{x = \pm 1\}$. Explain why this cannot hold for every point in the cube with $\{z = 1\}$.

3. Define the general notion of a Poincaré map of a flow between two cross sections.

4. Calculate the Poincaré map between the top of the cube $\{z = 1\}$ and the sides $\{x = \pm 1\}$.