UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2007

This paper is also taken for the relevant examination for the Associateship.

M3A22 / M4A22

MATHEMATICAL FINANCE

Date: Tuesday, 29th May 2007 Time: 2pm-4pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The change in price of an asset S in a time dt satisfies the stochastic differential equation

$$dS = a(S,t)dt + b(S,t)dX$$

where a and b are given functions of S and t and dX is sampled from a Brownian motion of mean zero and variance dt.

Derive the Black-Scholes equation for the fair price of an option $V(S,t)$ given that S satisfies the equation above and that r is the constant risk free interest rate.

Show that two special solutions of this equation are

$$V(S,t) = S$$

and

$$V(S,t) = e^{rt} \quad .$$

Also find $b(S,t)$ such that

$$V = e^{-\alpha t} S^3$$

is a solution where α is a specified constant.

2. (i) A vanilla call option $C(S,t)$ with exercise price E and exercise time T has payoff

$$C(S,T) = Max(S(T) - E, 0) \quad .$$

Describe the payoff for the corresponding vanilla European put option $P(S,t)$ and find a simple expression for

$$C(S,t) - P(S,t)$$

where both options have the same exercise time and exercise price.

(ii) A Binary call option $B_c(S,t)$ has payoff $B_c(S,T) = H(S-E)$ at time $t = T$. Define the payoff for the corresponding put option $B_p(S,t)$ and derive a simple relation between $B_c(S,t)$ and $B_p(S,t)$ for $t < T$.

(iii) Given that one can only buy and sell calls and puts of the above types with exercise prices 80, 90, 100 and 110 units and common exercise time T , construct a portfolio of options such that

$$V(S,T) = \begin{cases} 0 & S < 80 \\ 90 - S & 80 < S < 90 \\ 0 & 90 < S < 100 \\ S - 100 & 100 < S < 100 \\ 0 & S > 110 \end{cases} \quad .$$

3. A perpetual American Put has payoff

$$V(S) = Max\left(E^2 - S^2, 0\right)$$

Find the critical value $S = S^*$ at which it is optimal to exercise this option by maximising the price with respect to S^* . Hence deduce a condition on $\frac{\partial V}{\partial S}$ at this optimal free boundary (the asset pays no dividends, satisfies the usual equation $dS = \mu S dt + \sigma S dx$ and there is a constant interest rate r) .

4. (i) Show using Ito's lemma that for functions $F(t,X)$ for which the partial derivatives exist

$$\int_0^t \frac{\partial F}{\partial X}\,dX = F(t,X(t)) - F\left(0,X(0)\right) - \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)\,dt$$

and hence or otherwise show that one can write

$$\int_0^1 (1-t)\cos(2X(t))dX(t) = \int_0^1 (a+bt)\sin(2X(t))dt$$

if $X(t)$ is a Brownian motion with $X(0)=0$ and determine a and b .

(ii) If $S(0)=S_0$ and $X(0)=0$ solve the stochastic differential equation

$$dS = \mu S dt + \sigma S dX \quad .$$

(iii) With S defined in (ii) find a stochastic differential equation for $F=S^2$.

(iv) A binominal model assumes that an equity which initially has value S_0 can during a time step δt either rise to $u_1 S_0$ or fall to $v_1 S_0$ with equal probablility $\frac{1}{2}(u_1 > v_1)$. Determine u_1 and v_1 such that the mean and variance at step δt agree with that predicted by the equations in (ii) and (iii) above.

5. S_1 and S_2 are assets which satisfy differential equations of the form

$$\left.\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dX_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dX_2 \end{aligned}\right\} \qquad (1)$$

$\mu_1, \mu_2, \sigma_1, \sigma_2$ are constants and dX_1 and dX_2 increments of correlated Brownian motions $E(dX_1 dX_2) = pdt$.

Use a portfolio of the form $\pi = V(S_1, S_2) - \Delta_1 S_1 - \Delta_2 S_2$ to construct a partial differential equation for the fair price of an option $V(S_1, S_2, t)$.

Derive corresponding equations to (1) for the variables lnS_1, and lnS_2 and hence derive Ito's lemma for $V(F_1, F_2, t)$ with $F_1 = lnS_1, F_2 = lnS_2$.

Show by constructing an appropriate portfolio that a differential equation for the fair price for $V(F_1, F_2, t)$ can be obtained.