Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3A22/M4A22

Mathematical Finance

Date: Wednesday, 24th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

You should assume throughout that $dX(t)$ is a Brownian motion with

$$E\left(dX(t)\right) = 0\,,$$

$$E\left(\left(dX(t)\right)^2\right) = dt\,.$$

1. (a) Derive the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = 0$$

for the "fair" price of an option based on a security S which satisfies the stochastic differential equation

$$dS = a(S, t)dt + b(S, t)dX \, ,$$

where a and b are given functions of S and t, r is the risk free interest rate and D is the continuous dividend yield.

(b) A contract is agreed where the buyer will obtain the asset S at some time T in the future. Use the above equation to determine a fair price that the buyer should pay.

2. (a) A binomial model assumes that an equity which initially has value S can during a time step δt either rise to $u_1 S$ or fall to $v_1 S$, each with probability $\frac{1}{2}$. Beginning with a value S_0 at time $t = 0$, construct a binomial tree up to time $2\delta t$.

(b) Assuming the above binomial model, construct a portfolio at time t consisting of one option and a short position in a quantity $\triangle$ of the underlying. Thus at time t this portfolio has value

$$\pi = V - \triangle S,$$

where the value of V is to be determined. Given a risk free interest rate r (assumed constant) use a no-arbitrage argument to determine the value $V(s, t)$ of the option at time t in terms of option values at time step $t + \delta t$.

(c) Hence construct the price of a European Binary Call option at times $t = 0$ and $t = \delta t$ and at values of S in the binomial tree given an exercise time of $T = 2\delta t$ and a payoff

$$V(S, T) = H(S - E),$$

where

$$v_1^2 S_0 < E < u_1 v_1 S_0$$

and assuming a zero interest rate $(r = 0)$.

(d) Construct also the price for a Binary European Put, payoff

$$V(S, T) = H(E - S)$$

with $E = u_1 v_1 S_0$ and $r = 0$, at the same times and S values as the call, and check whether a form of Put-Call parity is true for these solutions.

3. (a) Find the solution of the stochastic differential equation

$$dS = \mu S dt + \sigma S dX$$

where μ and σ are constants and

$$X(0) = 0, \quad S(0) = S_0 \,.$$

(b) Evaluate

$$\int_0^1 X^2(\tau) dX(\tau) + \int_0^1 X(\tau) d\tau$$

when $X(0) = 0$.

(c) Solve

$$dS = \mu(S+1)dt + \sigma(S+1)dX \,,$$

where μ, σ are constants and

$$X(0) = 0, \quad S(0) = S_0 \,.$$

(d) If S satisfies the stochastic differential equation given in (a) above, derive the corresponding equation for

$$F = S^3 \,.$$

4. A European average strike call option has payoff

$$\max\ \left(S - \frac{1}{T}\int_0^T S(\tau)\,d\tau,\,0\right)$$

at time $t = T$.

(a) Defining

$$I(t) = \int_0^t S(\tau)d\tau$$

and

$$R(t) = \frac{S(t)}{\int_0^t S(\tau)d\tau} = \frac{S(t)}{I(t)}\ ,$$

derive equations for dI in terms of $S(t)$ and dt and for dR in terms of dS and dt, with coefficients involving I and R.

(b) In terms of I and R the payoff for the above option equals

$$I\max\left(R - \frac{1}{T},\,0\right) \quad \text{at } t = T\,.$$

The asset S follows the stochastic differential equation

$$dS = \mu S dt + \sigma S dX\,.$$

Writing the option value V as

$$V(S,R,t) = IW(R,t),$$

use the usual Black-Scholes hedging argument to deduce a partial differential equation for $W(R,t)$ of the form

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + a(r,R)\frac{\partial W}{\partial R} + b(r,R)W = 0,$$

and determine the functions $a(r,R)$ and $b(r,R)$. You may use an alternative argument beginning with $V(S,I,t)$ if you prefer.

5. The interest rate r is assumed to be satisfied by a stochastic differential equation

$$dr = \sigma(t) dX \, .$$

(a) By hedging with a bond of a different maturity, derive the bond pricing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 V}{\partial r^2} - \lambda\frac{\partial V}{\partial r} - rV = 0, \qquad (1)$$

where $\lambda(r,t)$ is an arbitrary function.

(b) Assuming that λ is a function of t only, and that a bond has payoff at maturity $t = T$ of one unit, i.e.

$$V(r,T) = 1 \, ,$$

find a solution of (1) of the form

$$V(r,t) = \exp\{A(t) + B(t)r\},$$

where $A(t)$ can be written

$$A(t) = -\int_t^T \left[\lambda(t')(t'-T) + \alpha(t')(t'-T)^2\right] dt'$$

and determine the function $\alpha(t')$.

(c) If at time t_0 bond prices are given for a continuous range of maturities, T, so that $V(r,t_0;T)$ is known as a function of T, determine $\lambda(T)$ in terms of $\dfrac{\partial^2}{\partial T^2}\,(\ln V(r,t_0;T))$.