Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

# M3A22/M4A22 Mathematical Finance

Date: Tuesday, 24th May 2005 Time: 10 am – 12 noon

Throughout the paper you may assume that $dX$ is an increment of a Brownian Motion (Wiener process) with

$$E(dX) = 0 \qquad \text{and} \qquad E(dX^2) = dt$$

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Derive the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right)$$

for the "fair" price of an option based on a security $S$ which satisfies the stochastic differential equation

$$dS = a(S,t)\,dt + b(S,t)\,dX$$

where $a$ and $b$ are given functions of $S$ and $t$, and $r$ is the risk-free interest rate.

(ii) A contract has payoff at time $t = T$ given by

$$V(S,T) = \begin{cases} E_1 - S, & \text{if} \quad S \le SE_1\,, \\ 0, & \text{if} \quad E_1 \le S \le E_2\,, \\ S - E_2, & \text{if} \quad S \ge E_2\,. \end{cases}$$

(a) State how this can be guaranteed by buying calls and puts at time $t$.

(b) How can we guarantee the payoff given that we can only

- buy calls,
- have money in the bank at constant interest rate $r$, and
- sell the asset?

2. An asset $S$ with value $a$ today follows the Binomial tree

$$\begin{array}{cccc} & & & a+20 \\ & & a+10 & \\ & a & & a \\ & & a-10 & \\ & & & a-20 \\ & & & \\ \text{Time} & 0 & T_1 & T \end{array}$$

Construct the corresponding option pricing tree

$$\begin{array}{cccc} & & & V_2 \\ & & V_1 & \\ & V & & V_0 \\ & & V_{-1} & \\ & & & V_{-2} \\ & & & \\ \text{Time} & 0 & T_1 & T \end{array}$$

for the cases

(i) European Call payoff $\quad V(S,T) = \max(S-a-5,\, 0)$

(ii) European Put payoff $\quad V(S,T) = \max(a-5-S,\, 0)$

(iii) Hence of otherwise price an option with payoff

$$V(S,T) = \begin{cases} S-a-5 & S \geq a+5 \\ 0 & a-5 \leq S \leq a+5 \\ a-5-S & S \leq a-5 \end{cases}$$

It is assumed that the risk-free interest rate is zero.

3. (i) Solve the stochastic differential equation

$$dS = aS\,dt + bS\,dX$$

where $a$ and $b$ are constants, and $S(0) = S_0$, $X(0) = 0$.

(ii) Show that the solution of

$$dS = 2a^2\,(\tan S)\,(\sec^2 S)\,dt \;+\; 2a\,(\sec S)\,dX$$

can be written

$$S = \arcsin(\alpha X(t) + \sin S_0)$$

where $S_0 = S(0)$, $X(0) = 0$, $a$ is a constant, and $\alpha$ is a constant to be determined.

(iii) By considering the form $S(t) = (A + X/\alpha)^{\alpha}$ or otherwise, solve

$$dS = \frac{1}{3}S^{1/3}dt + S^{2/3}dX$$

for the two cases

(a) $S(0) = 1$, and

(b) $S(0) = 0$,

with $X(0) = 0$ in each case.

4. Two assets $S_1$ and $S_2$ satisfy the stochastic differential equations

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dX_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dX_2$$

and the Brownian Motions $dX_1$ and $dX_2$ are correlated so that $E(dX_1\, dX_2) = p\, dt$ for $-1 < p < 1$. Derive Ito's lemma

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}S_1^2\sigma_1^2\frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}S_2^2\sigma_2^2\frac{\partial^2 V}{\partial S_2^2} + pS_1S_2\sigma_1\sigma_2\frac{\partial^2 V}{\partial S_1 \partial S_2}\right) dt + \frac{\partial V}{\partial S_1}dS_1 + \frac{\partial V}{\partial S_2}dS_2$$

for a function $V(S_1, S_2, t)$.

Use this and a portfolio of the form

$$\Pi = V - \Delta S_1 - \Delta_1 V_1\,,$$

where $V_1$ is another option, to derive an equation which the "fair" option price must satisfy.

Use the fact that both $S_1$ and $S_2$ must satisfy this equation to deduce the Black-Scholes differential equation for an option on two assets $S_1$ and $S_2$.

5. (i) Indicate briefly how the Bond pricing equation

$$\frac{\partial B}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 B}{\partial r^2} + (u - \lambda w)\frac{\partial B}{\partial r} - rB = 0$$

is derived, where

$$dr = (u - \lambda w)dt + w dX$$

is the risk-neutral model for the interest rate.

(ii) If the risk-neutral model is

$$dr = ar^2 dt + br^{3/2} dX,$$

find the price of a "perpetual" bond whose value is $\max(r - E, 0)$ and which can be exercised at any time, where $E > 0$.