Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M3/4A22 Mathematical Finance

Date: Friday, 21st May 2004 Time: 10 am –12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Throughout the paper you may assume that dX is an increment of a Brownian Motion (Weiner process) with

$$E(dX) = 0$$

$$E(dX^2) = dt$$

Calculators may not be used.

Statistical tables will not be available.

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1. A Vanilla European call option has 'fair' price $V_c(s,t;\ E,T)$ and a Vanilla European Put option $V_p(s,t;\ K,T)$.

   Define the contracts and pay offs of these two options.

   A contract $P,(s,t)$ has at time T a pay off of the following form:-

$$P_1(S,\,T)\begin{cases} = \quad E_1 - S & 0 < S < E_1\ , \\ \\ = \quad 0 & E_1 < S < E_2\ , \\ \\ = \quad S - E_2 & E_2 < S < \infty\ . \end{cases}$$

   Decompose this contract into one involving Vanilla Calls and Puts and hence price $P_1(s,t)$.

   How would one price this contract at time t if the only Option available in the market place was a Vanilla European Call?

   Use similar arguments to price the contract $P_2(s,t)$ where

$$P_2(s,\,T) = \begin{cases} E_1 - s + A & 0 < s < E_1\ , \\ \\ A & E_1 < s < E_2\ , \\ \\ s - E_2 + A & E_2 < s < \infty\ . \end{cases}$$

   (The interest rate r is constant and the asset S is traded).

2. (i) If in a time step δt an asset S either rises to a price S^+ or falls to a price S^- and an option price V changes to a price V^+ or V^- as the corresponding asset price changes, then show by constructing a risk free portfolio $V - \Delta S$ that

$$\Delta = \frac{V^+ - V^-}{S^+ - S^-},$$

and deduce a formula for $V(s,t)$ given that the spot interest rate is zero.

Show that this formula can be expressed in the form

$$V \equiv \frac{a(s - s^-)}{(s^+ - s^-)} + \frac{b(s^+ - s^-)}{(s^+ - s^-)}$$

and find a and b.

(ii) The Binomial tree for an asset S is as shown below

				140
			130	
		120		120
	110		110	
100		100		100
	90		90	
		80		80
			70	
				60
$t = 0$	$\frac{T}{4}$	$\frac{T}{2}$	$\frac{3T}{4}$	T

(i.e. the asset rises or falls by an amount 10).

(iii) Using the formula deduced in part (i), construct a tree corresponding to that in part (ii) above, for the option price V for..

(a) A European Binary Put option with payoff $H(110 - s)$ at time T (H is the Heaviside step function) and

(b) What would be the price at $t = 0$ of a European Binary Call option with the same strike as (a).

(c) Price (a) when there is a Barrier such that $V(110, t) = 0$.

3. (i) Use Ito's Lemma to deduce the following formula for stochastic differential equations and stochastic (Ito) integrals

$$\int_0^t \frac{\partial F}{\partial X}\, dX(s) = F \quad (X(t),t) - F(X(0),0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\,\frac{\partial^2 F}{\partial X^2} \right) ds$$

for a function $F(X(s),s)$ where $dX(s)$ is an increment of a Brownian motion.

(ii) Show that $F = \cos(aX(t) + b)$ is a solution of the stochastic differential equation

$$dF = -a\sqrt{(1-F^2)}dX \quad \frac{-a^2 F}{2}\, dt$$

where a and b are constants.

(iii) Solve the stochastic differential equation

$$dF = -\frac{1}{2}a^2 F\, dt + a\sqrt{(1-F^2)}\, dX$$

with $F(X(0)) = 0, \;\; X(0) = 0$

(iv) If

$$dF = \mu(F+1)\, dt + \sigma(F+1)\, dX$$

find F if $X(0) = 0$ and $F(0) = 0$.

4. (i) Given that an asset S satisfies the stochastic differential equation

$$dS = \mu S\, dt + \sigma S dX$$

find the stochastic differential equation satisfied by

$$x = lnS$$

Hence deduce Ito's lemma for $V(x,t)$

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\, \sigma^2 \frac{\partial^2 V}{\partial x^2} \right)\, dt + \frac{\partial V}{\partial x}\, dx.$$

and derive a partial differential equation for $V(x,t)$ by constructing a risk free portfolio by heading with the asset S and using the principle of no-arbitrage.

(ii) A shout option has pay-off $Max(S - E, 0)$ at expiry $(t = T)$.

The buyer is allowed one shout before expiry. If he shouts at time $t_1 < T$ where $S(t_1) > E$ he receives $S(t_1) - E$ at t_1 and an additional $Max(S(T) - S(t_1), 0)$ at time T. His pay off if he shouts is thus

$$V_2 = (S(t_1) - E)\, e^{r(T-t_1)} + Max\ (S(T) - S(t_1), 0)$$

and if he doesn't shout it is

$$V_1 = max\ (S(T) - E, 0)$$

(r is the risk free interest rate).

If the share price rises above E before expiry, should the buyer shout? Give arguments to justify your answer.

5. The interest rate r is assumed to be governed by a stochastic differential equation of the form

$$dr = u(r,t)\,dt + w(r,t)\,dX$$

By hedging with a bond of different maturity derive the bond pricing equation for a zero coupon bond

$$\frac{\partial V}{\partial t} + \frac{1}{2}\,w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0$$

where $\lambda(r,t)$ is an arbitrary function.

Given a model where both u and λ are zero, calculate the price $V(r,t)$ for a bond paying unity at time T for the following cases:-

(i) $w = w_0$ constant for all time.

(ii) $w = \frac{-w_0(t-T)}{T}$

(iii) $w = \frac{-2w_0(t-T)}{T}$