1. Consider the population dynamics of species $x \geq 0$:

$$\dot{x} = rx - x(1 - x)$$

(a) Sketch the vector fields that occur for $r = 0$ and $r = 1$.

(b) Show that a saddle-node bifurcation occurs at a critical value of r, to be determined.

(c) Sketch the bifurcation diagram of fixed points x^* versus r.

(d) Analytically solve this ODE. Determine for what value of t the solution blows up (you may assume $r > 1$).

2. Consider the predator-prey system

$$\dot{x} = x[x(1 - x) - y] \qquad \dot{y} = y(x - a)$$

where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and $1 > a > 0$ is a control parameter.

(a) Sketch the nullclines in the first quadrant, $x, y \geq 0$. Show that the fixed points are $(0, 0), (1, 0)$ and $(a, a - a^2)$, and classify the fixed point $(1, 0)$.

(b) Define a supercritical Hopf bifurcation for a 1-parameter family of systems of 2 first-order ODEs. Assuming a supercritical Hopf bifurcation takes place at the parameter value a_c, sketch the topologically different phase portraits for $a < a_c,\ a = a_c$ and $a > a_c$, near the equilibrium point.

(c) For the fixed point $(a, a - a^2)$, calculate the Jacobian and its eigenvalues. Hence determine the unique value of $a \in (0, 1),\ a_c$, where a Hopf bifurcation could occur.

(d) Estimate the frequency of limit cycle oscillations for a near the bifurcation.

3. Let J be the molecular bond graph below left, and K the molecular bond graph below right.

(a) Determine whether J and K have the same abstract graphs or not, and also whether there is a homeomorphism from the embedded graph J to the embedded graph K. (You need only to show the deformation, not prove continuity, etc. directly.)

(b) Show that if an embedded graph G is topologically achiral in $\mathbb{R}^3$ then it is topologically achiral in S^3.

(b) Let h be a homeomorphism from $\mathbb{R}^3$ to itself or from S^3 to itself. If h is isotopic to a reflection map, then h is an *orientation-reversing*, or *o-r* homeomorphism. If h is isotopic to the identity map, it is an *orientation-preserving* or *o-p* homeomorphism. Every homeomorphism of $\mathbb{R}^3$ or S^3 is either o-r or o-p.

Using this, one could alternatively define a molecular bond graph $G \subseteq \mathbb{R}^3$ to be topologically achiral if there exists an o-r homeomorphism of $(\mathbb{R}^3, G)$ taking G to itself. Prove that this definition of topological achirality is equivalent to the standard one.

(You may assume that the composition of an o-p homeomorphism with an o-r homeomorphism is an o-r homeomorphism, while the composition of 2 o-r homeomorphisms is an o-p homeomorphism.)

4. A discrete model for population growth is given by

$$x_{n+1} = H_{r,b}(x_n) = r x_n (1 + x_n)^{-b}$$

for $r > 0$ and $b > 1$. For $x > 0$, $H_{r,b} > 0$ so, unlike the logistic model, the iterates of an initial condition $x_0 > 0$ never become negative.

(a) Assume $r > 1$. What is the non-zero fixed point, x^*? If $1 < b < 2$, what type of fixed point is x^*?

(b) Show that there is a unique fixed point for $r < 1$ and prove it is globally attracting.

(c) Determine the limiting behaviour of $H_{r,b}$ and $\frac{H_{r,b}}{x}$ as $x \to \infty$. Then find the unique critical point x_c where $H'_{r,b} = 0$.

(d) For a given function $f(x)$, the *Schwarzian derivative* is given by

$$S_f(x) = \frac{f'''(x)f'(x) - 3/2 f''(x)^2}{f'(x)^2}$$

.

It can be proven that if the Schwarzian derivative is negative and there exists only 1 critical point, then there is a unique periodic sink. Show that there is only 1 periodic sink $\forall b \geq 2$, and interpret this biologically.

(You may assume that the basin of attraction of x^* is all $x > 0$.)

5. During the fundamental biochemical process of glycolysis, living cells obtain energy by breaking down sugar. It can often proceed in an oscillatory fashion. In dimensionless form, a simple model of the oscillations is:

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

where x and y are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate) and $a, b > 0$ are kinetic parameters.

(a) Sketch the nullclines (roughly), and determine the direction of the trajectories on these nullclines.

(b) Consider the trapezoidal region R with edges the x-axis, the y-axis from 0 to b/a, the horizontal line joining $(0, b/a)$ to $(b, b/a)$, and the diagonal line of slope -1 extending from the point $(b, b/a)$ to the y-nullcline. Determine the direction of the trajectories on the 4 edges of this region. (Hint: For the diagonal line, consider $\dot{y}/\dot{x} = dy/dx$ for large x, and look at $\dot{x} - (-\dot{y})$ to determine when dy/dx is more negative than -1 $=$ slope of the diagonal line.)

(c) Determine a and b in terms of the trace, when the fixed point inside R is a repeller.

(d) State the Poincaré-Bendixson theorem, and hence show there is a closed orbit in R.