Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3A21/M4A21

Mathematical Biology

Date: Tuesday, 23rd May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider the model

$$\frac{\partial u}{\partial t} = r\left(1 - \frac{u}{K}\right)u - \frac{u}{1+u} + \frac{\partial^2 u}{\partial x^2}$$

for a population subject to predation and diffusion, where $r > 0$, $K > 0$.

(i) Find the steady state equation, and compute the three constant steady states $u = u_i$, $i = 1, 2, 3$. Determine condition(s) on r in terms of K such that the constant steady states are real, distinct and positive (including 0), and choose the indices such that $u_1 < u_2 < u_3$. Show that these conditions give an open interval of admissible r for any $K \neq 1$. Assuming these conditions hold, sketch a phase portrait for the steady state equation in the (u, v)-plane, where $v = u_x$. (**Hint**: use that the equation is Hamiltonian.) In case there are multiple saddles, you do *not* have to determine the order of their level sets.

(ii) Derive a suitable system of ODE's to describe wave solutions travelling with constant speed c ($c > 0$). Take parameter values $K = 4$, $r = 16/21$ and analyze the stability of the three equilibrium points. Sketch a phase portrait and show that there exists a wave solution with $\lim_{x\to-\infty} u(x,t) = u_1$ and $\lim_{x\to+\infty} u(x,t) = u_2$, and a wave solution with $\lim_{x\to-\infty} u(x,t) = u_3$ and $\lim_{x\to+\infty} u(x,t) = u_2$.

2. Consider the enzyme reaction

$$A + B + E \underset{k_1}{\overset{k_2}{\leftrightarrows}} C \underset{k_3}{\rightarrow} P + E,$$

where E is the enzyme, and P is the final product.

(i) Denote the concentrations of the chemicals involved by $a(t)$, $b(t)$, $e(t)$, $c(t)$, $p(t)$ and use the law of Mass Action to derive a system of ODE's describing this equation (in a spatially independent setting). $a(0) = a_0$, $b(0) = b_0$, $c(0) = 0$, $e(0) = e_0$ and $p(0) = 0$. Show that $e(t) + c(t)$ and $a(t) - b(t)$ are constant and equal to e_0 and $d = a_0 - b_0$, respectively. Use this to reduce the system to a system of (two) ODE's for b and c.

(ii) Assume that $e_0 \ll b_0$, and apply a rescaling $t \mapsto \tau$, $b \mapsto u$, $c \mapsto v$ to get the fast system

$$\frac{1}{\epsilon}\frac{\mathrm{d}u}{\mathrm{d}\tau} = \alpha v - (u+\delta)u(1-v)\ , \ \frac{\mathrm{d}v}{\mathrm{d}\tau} = (u+\delta)u(1-v) - (\alpha+\beta)v$$

where $\alpha = k_2/(k_1 b_0^2)$, $\beta = k_3/(k_1 b_0^2)$, $\delta = d/b_0$, and $\epsilon = e_0/b_0$. Rescale time once more to obtain a slow system. Determine approximate solutions for both systems, and give the timescales on which these solutions hold. Use this to sketch an approximate solution in the u, v plane.

3. Consider the following system that potentially exhibits pattern formation via the Turing mechanism:

$$u_t = u^2 - uv + \nabla^2 u \ , \ v_t = -v + 3uv - 2v^2 + d\nabla^2 v.$$

where $d > 0$ is a parameter, $u = u(r,t)$ and $v = v(r,t)$ are the concentrations of the chemicals involved, and a two-colour pattern arises depending on the concentration of u. We take $r = (x,y)$ on some domain $D \subset \mathbb{R}^2$, with $\nabla u \cdot n = \nabla v \cdot n = 0$ on the boundary of D. Here n is normal to the boundary.

(i) Compute the equilibria of the spatially independent system in $u \geq 0$, $v \geq 0$, and determine the equilibrium (u_0, v_0) that is linearly stable under spatially independent perturbations.

(ii) Derive a condition on d such that the system is linearly unstable at (u_0, v_0) under spatially dependent perturbations. Please note that just *stating* the condition will not suffice, you need to present a *derivation*.

4. We consider an SIR model where the infected class can either return to the susceptible class, or move on to the removed class:

$$\frac{\mathrm{d}\bar{S}}{\mathrm{d}\bar{t}} = -a\bar{S}\bar{I} + b\bar{I} \ , \ \frac{\mathrm{d}\bar{I}}{\mathrm{d}\bar{t}} = a\bar{S}\bar{I} - b\bar{I} - c\bar{I} \ , \ \frac{\mathrm{d}\bar{R}}{\mathrm{d}\bar{t}} = c\bar{I},$$

with $\bar{S}(0) = \bar{S}_0 > 0$, $I(\bar{0}) = \bar{I}_0 \geq 0$, $\bar{R}(0) = 0$, and a, b, c are strictly positive parameters.

(i) Apply a suitable rescaling to $\bar{S}$, $\bar{I}$, $\bar{R}$ and/or $\bar{t}$ to obtain the system

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -SI + \beta I \ , \ \frac{\mathrm{d}I}{\mathrm{d}t} = SI - \beta I - \gamma I \ , \ \frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I,$$

with $S(0) = 1$, $I(0) = I_0$ and $R(0) = 0$. Express β, γ and I_0 in terms of the original parameters and show that the total population size is constant.

(ii) Compute the equilibria and determine their stability. Sketch a phase portrait in the (S, I) plane. Determine the regions in the phase plane where S is monotone (increasing or decreasing) along solutions. Assuming that $S(0) = 1$ is in one of these regions, derive an equation for $\frac{\mathrm{d}I}{\mathrm{d}S}$ and use it to obtain an equation for $S_\infty = \lim_{t\to+\infty} S(t)$ (you do not have to solve this equation).

5. Consider the equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\delta u(t) - (1-\delta)u(t-T),$$

where $T \geq 0$ is the delay time, and $\delta \in [0, 1)$.

(i) What is the stability of the equilibrium $u = 0$ for $T = 0$? Show that there exists a $T_* = T_*(\delta) > 0$ where the stability changes if and only if $\delta < 1/2$. Show that $T_* \to +\infty$ as δ converges to $\frac{1}{2}$ from below. (**Hint**: Try solutions of the type $e^{\lambda t}$.)

(ii) Now take $\delta = 0$. Let $T = T_*(0) + \epsilon$ for ϵ positive and small. Let $u(t) = e^{\lambda t}$ be a solution of the delay equation of the form $\lambda = \mu + i\omega$ with $\mu = \mu_1 \epsilon + O(\epsilon^2)$ and $\omega = \omega_0 + \omega_1 \epsilon + O(\epsilon^2)$. Compute μ_1, ω_0 and ω_1. Does this solution show instability of the equilibrium $u = 0$ for T (slightly) larger than $T_*(0)$? Explain your answer.