UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION (MATHEMATICS) MAY – JUNE 2005
This paper is also taken for the relevant examination for the Associateship

M3A21 MATHEMATICAL BIOLOGY

DATE: Wednesday, 31 May 2005 TIME: 2.00 pm–4.00 pm

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) A bacterial organism has a population density $U(X, Y, T)$ that obeys the partial differential equation

$$U_T = rU\left(1 - \frac{U}{N}\right) + D(U_{XX} + U_{YY}) \qquad (1)$$

where D, r and N are constants, explain what each term on the right-hand side of (1) biologically represents.

(b) Rescale the variables appropriately to deduce the non-dimensional equation

$$u_t = u(1 - u) + u_{xx} + u_{yy}. \qquad (2)$$

This equation holds in a planar rectangular petri dish $-L/2 \le x \le L/2$, $-H/2 \le y \le H/2$. On the sides $y = \pm H/2$, $-L/2 \le x \le L/2$ there lies an antibiotic that instantly kills any of the organism that touches it. On the sides $x = \pm L/2$, $-H/2 \le y \le H/2$ there is a glass wall across which no organism can cross.

Assign boundary conditions for u along the edges of the rectangular dish. At $t = 0$ the population level is very low, find an appropriate criterion for the population to grow.

Now replace the glass walls along $x = \pm L/2$, $-H/2 \le y \le H/2$ with the antibiotic boundary condition and deduce the criterion for the population to grow.

In this second scenario, you are requested to design an experiment to deliberately and quickly grow the organism using rectangular petri dishes of fixed area, A, show that square petri dishes will be better than rectangular dishes of the same area.

2. Two chemicals $u(x,y,t), v(x,y,t)$ co-exist inside a rectangular region $0 \leq x \leq 3\pi$ and $0 \leq y \leq 2\pi$ and satisfy the coupled non-dimensional reaction diffusion equations

$$u_t = u(1+u-2v) + \nabla^2 u, \quad v_t = v(1+2u-3v) + d\nabla^2 v,$$

where d is a constant and ∇^2 is the two-dimensional Laplacian. Additionally, on the boundary of the region no-flux boundary conditions hold. Write down the conditions for instability (you need not derive them) and show that these equations can potentially sustain a diffusion-driven (Turing) instability. Determine the uniform state about which the instability can occur and find conditions upon d that will lead to this instability and, if $d = 10$, determine the band of unstable wavenumbers associated with the instability.

Let the uniform state be $(\hat{u}, \hat{v})$. Then, if

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} + \epsilon \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} e^{\lambda t} \phi$$

for constants u_0, v_0 and growth rate $\lambda(k^2)$, with eigenfunctions $\phi(x,y)$ and wavenumbers k^2, find the spatial eigenfunctions associated with the unstable range of wavenumbers determined above. Describe how you would determine which one is preferred (you are not expected to find the preferred state).

Turn over...

3. Consider the following model for the evolution of a population U

$$\frac{dU}{dT} = rU\left(1 - \frac{U}{K}\right)(U - U_c) - \epsilon U \qquad (3)$$

with r, K, U_c and ϵ as positive constants, and with the constraint that $U_c < K$. Rescale the variables in this model to obtain the non-dimensional equation

$$\frac{du}{dt} = u(1-u)(u-c) - eu$$

and express c, e in terms of r, K, U_c, ϵ.

Find the fixed points of this system assuming $0 < c < 1$ and discuss their existence and stability. Pay particular attention to the biological significance of the stability of these fixed points, and find the maximal value of e beyond which extinction occurs.

Using a sketch of $u(1-u)(u-c)$ and eu, or otherwise, discuss how you would expect the population to vary if initially $u = 1$ and $e = 0$ and then the parameter e is increased to a high value. What is the minimal (non-zero) sustainable population?

Set $e = 0$ in the model and discuss what will happen if $u(0) > c$ or $u(0) < c$.

Given your findings about this model, return to equation (3) and interpret the biological significance of the terms within that equation.

4. Consider an enzyme E that reacts with a substrate S undergoes a two stage process, with complex's C_1, C_2 before creating a product P according to the chemical equations

$$S + E \underset{k}{\overset{k_1}{\rightleftharpoons}} C_1, \qquad S + C_1 \overset{k}{\rightharpoonup} C_2 \overset{k_2}{\rightharpoonup} C_1 + P. \tag{4}$$

The concentrations of E, S, C_1, C_2, P are e, s, c_1, c_2, p and at $t = 0$ are $e_0, s_0, 0, 0, 0$.

(a) Using the law of mass action deduce equations for the evolution of s, c_1, c_2, e and p. Carefully explain the Michaelis-Menten steady state hypothesis, and then deduce that the reaction velocity dp/dt is given by

$$\frac{k_2 e_0 s^2}{K_1 K_2 + K_2 s + s^2}$$

where $K_1 = k/k_1$ and $K_2 = k_2/k$.

(b) Now consider a slow/fast reaction rate limit: $k \to \infty$, such that $K_1 \to \infty$, $K_2 \to 0$ but their product $K_1 K_2 \to \beta$ where $\beta = k_2/k_1$ is a constant. Write down the reaction velocity in this limit.

Then deduce that for large times

$$s \sim s_0 \exp\left(\frac{s_0^2}{2} - k_1 \beta e_0 t\right).$$

5. State the Law of Mass Action.

(a) Consider three populations: susceptibles S, infectives I and a removed class R, with population densities s, i and r respectively. Use the law of mass action to deduce the "chemical" equations:

$$S + mI \overset{k_1}{\rightharpoonup} nI \;,\; I \overset{k_2}{\rightharpoonup} R \qquad (5)$$

(for some constants m, n, k_1, k_2 that you must determine) such that the population densities satisfy

$$\frac{ds}{dt} = -ais, \quad \frac{di}{dt} = ais - bi, \quad \frac{dr}{dt} = bi.$$

If the total population is n and given that at $t = 0$ $i = n - s_0$ and $s = s_0$, show that the total number of survivors after the infection has run its course is given by the root of

$$s - \frac{b}{a} \log \frac{s}{s_0} = n.$$

Show that an epidemic will occur if $s_0 a/b > 1$.

(b) A model for spatially mobile susceptibles and infectives is

$$\frac{\partial s}{\partial t} = -ais + D\frac{\partial^2 s}{\partial x^2}$$

$$\frac{\partial i}{\partial t} = ais - bi + D\frac{\partial^2 i}{\partial x^2}$$

consider travelling wave solutions in the form $s = s(x - ct) = s(\xi)$ and $i = i(x - ct) = i(\xi)$ with $s(\infty) = s_0$ and $i(\pm\infty) = 0$ and $s(-\infty) = s_\infty$ where s_∞ is unknown. By considering the behaviour of the solution in the neighbourhood of $s = s_0, i = 0$ and assuming that $s_\xi = 0, i_\xi = 0$ (the subscript ξ denotes the derivative with respect to ξ) as $\xi \to \pm\infty$, deduce that there is a minimum allowable wavespeed c and show that it is given by

$$c = 2(as_0 D)^{\frac{1}{2}} \left(1 - \frac{b}{as_0}\right)^{\frac{1}{2}}.$$