1. A two-dimensional point vortex of strength Γ is placed at the point (a, b) (with $a > 0$ and $b > 0$), and another point vortex of strength $-\Gamma$ is placed at the point $(a, -b)$. The fluid occupies the region $x > 0$, and there is a solid vertical wall along the y axis. Assume that the flow is steady, incompressible, and irrotational.

(a) Write down the complex potential $\Omega(z)$ for this flow.

Hint: You can put image vortices inside the wall so as to satisfy the required boundary condition at the wall.

(b) What is the fluid velocity at the point $(a, 0)$?

(Make sure you give separate expressions for the real components u and v of the velocity.)

(c) What is the total force $\boldsymbol{F} = (F_x, F_y)$ on the wall?

Hint: You may use the Blasius formula $F_x - i\,F_y = \frac{1}{2} i\rho \int_C \left(\frac{d\Omega}{dz}\right)^2 dz$, *where* ρ *is the fluid density, and* C *is a closed contour encircling the solid body.*

2. Consider the unsteady flow given, in Cartesian coordinates, by

$$(u, v) = t^2 \left(x^2 y\,,\ -xy^2\right)$$

where u and v are the x and y components of the velocity, respectively, and t is time.

(a) Does this velocity field admit a streamfunction? If so, what is it?

(b) Find formulas for the instantaneous streamlines associated with this flow and verify that this streamline distribution remains the same at all times.

(c) At $t = 0$ a tiny blob of dye is used to colour the fluid particle situated at the point $(1, 1)$. Find the subsequent position of this dyed fluid particle at time $t = 1$.

(d) What is the vorticity scalar associated with this velocity field?

3. An ideal fluid in three dimensions is undergoing steady rigid-body rotation at an angular velocity Ω. The axis of rotation is parallel to the gravitational force vector, which points in the $-\hat{\boldsymbol{k}}$ direction. The top surface of the fluid is free and passes through the origin of the coordinate system. Above the fluid is a gas at constant pressure p_{atm}. The fluid has constant density ρ.

(a) Write down the velocity field inside the fluid in cylindrical polar coordinates. Using the expression for the divergence of a vector field given below, verify that it is incompressible.

(b) Find the pressure distribution inside the fluid, and the shape of the free surface.

(c) Is the flow irrotational? Justify your answer.

The Euler equations governing the velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)$ of an ideal fluid of density ρ are given, in cylindrical polar coordinates, by

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_\theta u_r}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g
\end{aligned}$$

where

$$(\boldsymbol{u}\cdot\nabla) = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

and

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0\ .$$

4. A point vortex of strength Γ is located at $(x, y) = (0, a)$, and another point vortex of strength $-\Gamma$ is located at $(x, y) = (0, -a)$.

(a) Identifying the coordinates x and y with points in the complex plane, write down the complex potential for the combination of the two vortices.

(b) Letting $\Gamma = \mu/a$ and taking the limit $a \to 0$ in (a), find the complex potential for a dipole. What is the velocity field associated with the dipole, in cartesian coordinates?

(c) Assuming the two vortices influence each other, find the net motion of the two vortices (without taking the limit $a \to 0$).

5. A three-dimensional spherical bubble of gas, of radius $R(t)$, is surrounded by an ideal fluid of constant density ρ which extends out to infinity. The pressure of the fluid at infinity is p_∞. The total mass of gas in the bubble is M. The gas inside the bubble satisfies a polytropic equation of state relating its pressure p_b to its density ρ_b by

$$p_b = A\,\rho_b^\gamma$$

where A and γ are (known) constants, and $A > 0$. Gravity is negligible.

(a) Assuming that the flow is irrotational and is spherically symmetric, derive the ordinary differential equation satisfied by $R(t)$.
(Recall that $\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)$ for spherical symmetry.)

(b) In terms of the parameters given, find an expression for the equilibrium radius R_e of the gas bubble.

(c) If the radius of the bubble is perturbed from its equilibrium value, R_e, two types of behaviour can occur, depending on the value of γ (which we assume here can be positive or negative). Describe these in words, in terms of the change of pressure as the radius of the bubble changes.