Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3A2/M4A2

Inviscid Flow Theory

Date: Tuesday, 30th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

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1. An unsteady flow is modelled by the two-dimensional velocity field

$$(u, v) = -U_0 \sin t \left(e^{-x}, y\, e^{-x}\right)$$

where U_0 is a positive constant and t is time.

(a) Find a streamfunction for the flow, if it exists.

(b) Sketch the streamlines, showing the direction of the flow,
(i) For a typical time in the range $0 < t < \pi$,
(ii) For a typical time in the range $\pi < t < 2\pi$.

(c) Setting $U_0 = \frac{1}{4}$, find the path of the fluid particle that is situated at the point $(0, 1)$ at time $t = 0$. Sketch this particle path.

(d) At what points, if any, does the particle in (c) momentarily come to rest? Indicate these points on your sketch in (c).

2. A point vortex of circulation $\Gamma > 0$ is at $z = z_1(t)$ in the complex plane, and a second point vortex of circulation -2Γ is at $z = z_2(t)$. Each vortex moves under the effect of the other vortex's velocity field.

(a) Write down the ordinary differential equations satisfied by $z_1(t)$ and $z_2(t)$.

(b) Solve for the complex number representing the vector $\eta(t) = z_1(t) - z_2(t)$ between the two vortices. What is the rotation frequency of the vortices around each other? In what direction do they rotate?

(c) Find another constant of motion by taking a linear combination of the two complex equations in (a). Hence write down the full solution for $z_1(t)$ and $z_2(t)$.

3. A spherical bubble of gas, of radius $R(t)$, is surrounded by an ideal fluid of constant density ρ that extends out to infinity. The pressure and velocity of the fluid at infinity is taken to be zero. The gas inside the bubble is assumed to be an ideal gas with the following equation of state relating relating its pressure p_b, its density ρ_b, and its temperature T_b:

$$\frac{p_b}{\rho_b} = k\,T_b$$

where k is the ideal gas constant. The pressure p_b inside the bubble is assumed to be uniform. The total mass of the gas inside the bubble is M_b.

(a) Assuming that any flow that takes place starts from rest and is radially symmetric, derive the ordinary differential equation satisfied by $R(t)$.

(b) Let the initial radius be $R(0) = 1$ and assume the initial temperature and density of the gas are $T_b = T_{b0}$ and $\rho_b = \rho_{b0}$. Assuming that the gas evolves isothermally, show that the rate of increase of radius of the bubble when the radius reaches $R = 2$ is given by

$$\frac{1}{2}\sqrt{\frac{k\,T_{b0}\,\rho_{b0}\,\ln 2}{\rho}}$$

Hint: $\ddot{R} = \dfrac{1}{2}\dfrac{d}{dR}\left(\dot{R}^2\right)$.

4. A source of two-dimensional fluid with strength m is placed at the point (a, b) (with $a > 0$ and $b > 0$) and a sink of strength $-m$ at the point $(a, -b)$. There is a solid vertical wall along the y axis. Assume that the flow is steady, incompressible, and irrotational.

(a) Write down the complex potential $\Omega(z)$ for this flow.

Hint: You can put image sources and sinks inside the wall so as to satisfy the no-throughflow boundary condition at the wall.

(b) What is the fluid velocity at the point $(a, 0)$?

(Make sure you give separate expressions for the real components u and v of the velocity.)

(c) What is the total force $\boldsymbol{F} = (F_x, F_y)$ on the wall?

Hint: You may use the Blasius formula $F_x - i\,F_y = \frac{1}{2}i\rho \displaystyle\int_C \left(\frac{d\Omega}{dz}\right)^2 dz$ *, where* ρ *is the fluid density, and* C *is a closed contour encircling the solid body.*

5. A hurricane is modelled by an ideal fluid of constant density ρ with velocity field $\boldsymbol{u}$ given in plane polar coordinates (r, θ) as

$$\boldsymbol{u} = (0, u_\theta(r))$$

where

$$u_\theta(r) = \begin{cases} \frac{1}{2}\omega_0 r\,, & r \leq a; \\ \dfrac{\omega_0\, a^{\lambda+1}}{2r^\lambda}\,, & r > a; \end{cases}$$

where ω_0 is a constant, and a and λ are positive constants.

(a) Find the vorticity distribution associated with the flow. For what value of λ is the flow irrotational in the region $r > a$?

(b) Calculate the circulation around the circle $r = b$, where $b > a$.

(c) If the fluid pressure at infinity is zero, find the fluid pressure p_0 at the centre of the hurricane (i.e., at $r = 0$) as a function of ω_0, a, and λ. Sketch a graph of p_0 against λ for fixed ω_0 and a.

The Euler equations governing the two-dimensional velocity field $\boldsymbol{u} = (u_r, u_\theta)$ of an ideal fluid of density ρ are given, in plane polar coordinates, by

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_\theta u_r}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

where

$$(\boldsymbol{u}\cdot\nabla) = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}$$

and

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0\ .$$

The vorticity $\boldsymbol{\omega} = \omega\,\widehat{\boldsymbol{z}}$ of a two-dimensional velocity field $\boldsymbol{u} = (u_r, u_\theta)$ is given by the formula

$$\omega = \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)$$

where $\widehat{\boldsymbol{z}}$ is a unit vector along the z direction.