1.

(a) Show that the equation for the time evolution of an infinitesimal line segment $\delta\boldsymbol{r}$ being stretched by a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ is

$$\frac{D}{Dt}\,\delta\boldsymbol{r} = (\delta\boldsymbol{r}\cdot\nabla)\boldsymbol{u}\,.$$

(Make sure you give the definition of D/Dt.)

(b) Using the result in (a), derive an equation for the *length* of the infinitesimal segment (or its squared length). Write the result in terms of the rate-of-strain tensor (or matrix) and the vorticity tensor. Comment briefly on the respective role of these tensors in the resulting expression.

(c) Given the two-dimensional velocity field $\boldsymbol{u} = (\lambda x\,, -\lambda y)$, what is the length at time $t = 1$ of a straight segment initially between $(1, 0)$ and $(2, 0)$ at $t = 0$?

2. Consider the unsteady flow given, in Cartesian coordinates, by

$$(u, v) = \cos t \left(\frac{x^2}{2}\,,\ -xy \right)$$

where u and v are the x and y components of the velocity respectively and t is time.

(a) Does this velocity field admit a streamfunction? If so, what is it?

(b) Find formulas for the instantaneous streamlines associated with this flow and verify that this streamline distribution remains the same at all times.

(c) At $t = 0$ a tiny blob of dye is used to colour the fluid particle situated at the point $(1, 1)$. Find the subsequent position of this dyed fluid particle at time $t = 1$.

3. A three-dimensional spherical bubble of gas, of radius $R(t)$, is surrounded by an ideal fluid of constant density ρ which extends out to infinity. The pressure of the fluid at infinity is p_∞. The total mass of gas in the bubble is M. The gas inside the bubble satisfies a polytropic equation of state relating its pressure p_b to its density ρ_b by

$$p_b = A\,\rho_b^\gamma$$

where A and γ are (known) constants.

(a) Assuming that the flow is irrotational and is spherically symmetric, derive the ordinary differential equation satisfied by $R(t)$.

(Recall that $\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)$ for spherical symmetry.)

(b) In terms of the parameters given, find an expression for the equilibrium radius R_e of the gas bubble.

(c) Derive an expression for the frequency of small oscillations of the bubble about this equilibrium radius.

4. A point vortex of strength Γ is located at $(x, y) = (0, a)$, and another point vortex of strength $-\Gamma$ is located at $(x, y) = (0, -a)$.

(a) Identifying the coordinates x and y with points in the complex plane, write down the complex potential for the combination of the two vortices.

(b) Letting $\Gamma = \mu/a$ and taking the limit $a \to 0$ in (a), find the complex potential for a dipole. What is the velocity field associated with the dipole, in cartesian coordinates?

(c) Assuming the two vortices influence each other, what is the net motion of the dipole (without taking the limit $a \to 0$)?

5. Consider the following steady two-dimensional velocity field for an ideal fluid of constant density ρ, given in plane polar coordinates (r, θ):

$$\boldsymbol{u} = (0, u_\theta(r))$$

where

$$u_\theta(r) = \begin{cases} \omega_0 r^2\,, & r < a; \\ \frac{\omega_0\, a^3}{r}\,, & r \geq a; \end{cases}$$

where ω_0 is a constant and a is a positive constant.

(a) If the fluid pressure at infinity is p_∞, show that the fluid pressure at the core $(r = 0)$ is

$$p_\infty - \frac{3}{4}\,\rho\,\omega_0^2 a^4\ .$$

(b) Compute the vorticity associated with the flow.

(c) Define the total circulation Γ of the flow to be

$$\Gamma = \oint_{r=c} \boldsymbol{u} \cdot d\boldsymbol{\ell}$$

where the integration contour is a circle of radius $c > a$. Find a formula for the total circulation in terms of ω_0 and a.

(d) Use Stoke's theorem to compute Γ in terms of the vorticity.

The Euler equations governing the two-dimensional velocity field $\boldsymbol{u} = (u_r, u_\theta)$ of an ideal fluid of density ρ are given, in plane polar coordinates, by

$$\begin{aligned} \frac{\partial u_r}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{\partial u_\theta}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_\theta + \frac{u_\theta u_r}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \end{aligned}$$

where

$$(\boldsymbol{u} \cdot \nabla) = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}$$

and

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \ .$$

The vorticity $\boldsymbol{\omega} = \omega\,\hat{\boldsymbol{z}}$ of a two-dimensional velocity field $\boldsymbol{u} = (u_r, u_\theta)$ is given by the formula

$$\omega = \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)$$

where $\hat{\boldsymbol{z}}$ is a unit vector along the z direction.