1. The transformation in $\mathbb{R}^3$ from 3D Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) is given by the well-known formula

$$(x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$$

(a) Compute the 3-form $d^3x = dx \wedge dy \wedge dz$ in terms of (r, θ, ϕ) and $(dr, d\theta, d\phi)$.

In general, contraction of a vector field with a 3-form produces a 2-form $X \,\lrcorner\, d^3x = \mathbf{X} \cdot \hat{\mathbf{n}}\, dS$, where dS is the surface area element with unit normal vector $\hat{\mathbf{n}}$.

(b) Compute the 2-form $\beta = X \,\lrcorner\, d^3x$ obtained by substituting the vector field,

$$X = \mathbf{x} \cdot \nabla = x\partial_x + y\partial_y + z\partial_z = r\partial_r \,,$$

into the 3-form d^3x. Write the expression in both Cartesian and spherical coordinates.

(c) Is the 2-form β in (b) closed? Is it exact? Determine this by computing the 3-form arising from its exterior derivative, as

$$d\beta = d(X \,\lrcorner\, d^3x)$$

Express this 3-form in both Cartesian and spherical coordinates.

(d) Evaluate β in (c) on the spherical level surface $r = 1$. How is the result related to the geometry of a sphere? Is the 2-form β evaluated at $r = 1$ closed? Is it exact?

2. By using Cartan's formula, $\pounds_X \alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)$ and the two defining properties

$$\begin{aligned} X \,\lrcorner\, (\alpha \wedge \beta) &= (X \,\lrcorner\, \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \,\lrcorner\, \beta)\,, \\ d(\alpha \wedge \beta) &= (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta\,, \end{aligned}$$

prove the following three identities for Lie derivatives of a $k-$form α:

(a) $\pounds_{fX} \alpha = f \pounds_X \alpha + df \wedge (X \,\lrcorner\, \alpha)$

(b) $\pounds_X d\alpha = d\big(\pounds_X \alpha\big)$

(c) $\pounds_X (\alpha \wedge \beta) = (\pounds_X \alpha) \wedge \beta + \alpha \wedge \pounds_X \beta$

3. The canonical Poisson brackets are $\{q_k, p_m\} = \delta_{km}$ when expressed in phase space coordinates $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3$.

Given these canonical Poisson brackets, consider the following function for any $\boldsymbol{\xi} \in \mathbb{R}^3$,

$$J^\xi = \boldsymbol{\xi} \cdot (\mathbf{p} \times \mathbf{q}) \,.$$

(a) Compute the Poisson brackets $\{J^\xi, \mathbf{q}\}$ and $\{J^\xi, \mathbf{p}\}$ in vector form. Interpret these relations geometrically.

(b) Find the Hamiltonian vector field X_{J^ξ} for $J^\xi = \boldsymbol{\xi} \cdot (\mathbf{p} \times \mathbf{q})$.

(c) Find the functions of $\mathbf{q}$ and $\mathbf{p}$ that are left invariant by this vector field.

(d) Explain geometrically why these quantities are left invariant.

(e) Compute the evolution of Hamilton's canonical equations for the Hamiltonian

$$J^\xi = \boldsymbol{\xi} \cdot (\mathbf{p} \times \mathbf{q}) = p_1 q_2 - p_2 q_1 \,, \quad \text{when} \quad \boldsymbol{\xi} = (0, 0, 1)^T \,.$$

4. (a) Compute the Poisson brackets among

$$J_l = \epsilon_{lmn} p_m q_n \quad \text{for} \quad l, m, n = 1, 2, 3,$$

given the canonical Poisson brackets $\{q_k, p_m\} = \delta_{km}$.

(b) (i) Do the Poisson brackets $\{J_l, J_m\}$ close among themselves?

(ii) Write the Poisson bracket $\{F(\mathbf{J}), H(\mathbf{J})\}$ for the restriction of the dynamics to functions of $\mathbf{J} = (J_1, J_2, J_3)$.

(iii) Write in vector notation the dynamical equation $\dot{\mathbf{J}} = \{\mathbf{J}, H(\mathbf{J})\}$ for any Hamiltonian function $H(\mathbf{J})$.

(iv) Compute the dynamical equation for the Hamiltonian function

$$H(\mathbf{J}) = J^\xi = \boldsymbol{\xi} \cdot \mathbf{J}$$

for any vector $\boldsymbol{\xi} \in \mathbb{R}^3$. Interpret the solutions for this flow geometrically.

5. Recall the canonical Poisson bracket relations for oscillator variables on $\mathbb{C}^2$,

$$\{a_j, a_k^*\} = -\,2i\delta_{jk} \quad \text{and} \quad \{a_j^*, a_k\} = 2i\delta_{jk} \quad \text{where} \quad j, k = 1, 2$$

(a) Define the S^1-invariant quantities

$$\begin{aligned} R &= \frac{n}{2}\left|a_1\right|^2 + \frac{m}{2}\left|a_2\right|^2 \\ Z &= \frac{n}{2}\left|a_1\right|^2 - \frac{m}{2}\left|a_2\right|^2 \\ X + iY &= 2a_1^m a_2^{*n} \end{aligned}$$

Show that these variables are not independent by verifying that the function

$$C(X, Y, Z, R) = X^2 + Y^2 - 4\left(\frac{R+Z}{n}\right)^m \left(\frac{R-Z}{m}\right)^n$$

vanishes identically.

(b) Compute the Poisson brackets among R, X, Y and Z in Part (a).

(c) Use the Poisson brackets in Part (b) to write the Poisson bracket between two functions F and H of (X, Y, Z) as the triple vector product of gradients

$$\{F,\, H\} = -\,\nabla C \cdot \nabla F \times \nabla H\,, \quad \text{so that} \quad \{X,\, Y\} = -\,\partial C/\partial Z\,, \quad \text{etc.}$$

Hint: use $C(X, Y, Z, R) = 0$ from Part (a).

(d) Explain the geometric meaning of the equation of motion for this Poisson bracket. In particular, what is the orbit in $(X, Y, Z) \in \mathbb{R}^3$ when the Hamiltonian is chosen to be $H = Z$ for a given value of R?