Imperial College
London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3A16/M4A16

Dynamics II

Date: Wednesday 31st May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. In coordinates $(a_1, a_2) \in \mathbb{C}^2$, the Hopf map $\mathbb{C}^2/S^1 \to S^3 \to S^2$ is obtained by transforming to the four quadratic S^1 invariant quantities

$$(a_1, a_2) \to Q_{jk} = a_j a_k^* \,, \quad \text{with} \quad j, k = 1, 2\,.$$

Let the $\mathbb{C}^2$ coordinates be expressed as $a_j = q_j + ip_j$ in terms of canonically conjugate variables satisfying the fundamental Poisson brackets $\{q_k, p_m\} = \delta_{km}$, $k, m = 1, 2$.

(a) Compute all the Poisson brackets $\{a_j, a_k\}$ for $j, k = 1, 2$.

(b) Is the transformation $(q, p) \to (a, a^*)$ canonical? Explain why, or why not.

(c) Compute the Poisson brackets among the Q_{jk}, with $j, k = 1, 2$. State and prove whether they close among themselves and whether they are canonical?

(d) Make the linear change of variables,

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,.$$

Compute the Poisson brackets among the (X_0, X_1, X_2, X_3).

(e) Express the Poisson bracket $\{F(\mathbf{X}), H(\mathbf{X})\}$ in vector form among functions F and H of $\mathbf{X} = (X_1, X_2, X_3)$

(f) Show that the quadratic invariants (X_0, X_1, X_2, X_3) themselves satisfy a quadratic relation. How is this relevant to the Hopf map?

2. Consider canonical coordinates in phase space $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3$ with Poisson brackets $\{q_k, p_m\} = \delta_{km}$.

Given these canonical Poisson brackets, consider the following function for any $\boldsymbol{\xi} \in \mathbb{R}^3$,

$$J^\xi = \boldsymbol{\xi} \cdot (\mathbf{p} \times \mathbf{q})\,.$$

(a) Compute the Poisson brackets $\{J^\xi, \mathbf{q}\}$ and $\{J^\xi, \mathbf{p}\}$ in vector form. Interpret these relations geometrically.

(b) Find the Hamiltonian vector field X_{J^ξ} for $J^\xi = \boldsymbol{\xi} \cdot (\mathbf{p} \times \mathbf{q})$.

(c) Find the functions of $\mathbf{q}$ and $\mathbf{p}$ that are left invariant by this vector field.

(d) Explain geometrically why these quantities are left invariant.

(e) Compute the evolution of Hamilton's canonical equations for the Hamiltonian

$$J^{\hat{\mathbf{z}}} = \hat{\mathbf{z}} \cdot (\mathbf{p} \times \mathbf{q}) = p_1 q_2 - p_2 q_1\,.$$

3. (a) Compute the Poisson brackets among

$$J_l = \epsilon_{lmn} p_m q_n \quad \text{for} \quad l, m, n = 1, 2, 3,$$

given the canonical Poisson brackets $\{q_k, p_m\} = \delta_{km}$.

(b) (i) Do the Poisson brackets $\{J_l, J_m\}$ close among themselves?

(ii) Write the Poisson bracket $\{F(\mathbf{J}), H(\mathbf{J})\}$ for the restriction of the dynamics to functions of $\mathbf{J} = (J_1, J_2, J_3)$.

(iii) Write in vector notation the dynamical equation $\dot{\mathbf{J}} = \{\mathbf{J}, H(\mathbf{J})\}$ for any Hamiltonian function $H(\mathbf{J})$.

(iv) Compute the dynamical equation for the Hamiltonian function

$$H(\mathbf{J}) = J^{\xi} = \boldsymbol{\xi} \cdot \mathbf{J}$$

for any vector $\boldsymbol{\xi} \in \mathbb{R}^3$. Interpret the solutions for this flow geometrically.

4. By using Cartan's formula,

$$\pounds_X \alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)$$

prove the following identities for Lie derivatives for a $k-$form α:

(a) $\pounds_{fX} \alpha = f \pounds_X \alpha + df \wedge (X \,\lrcorner\, \alpha)$

(b) $\pounds_X d\alpha = d\big(\pounds_X \alpha\big)$

(c) $\pounds_X (X \,\lrcorner\, \alpha) = X \,\lrcorner\, \pounds_X \alpha$

5. The exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation

$$\begin{aligned} df &= f_{,j}\,dx^j =: \nabla f \cdot d\mathbf{x} \\ df \wedge dg &= f_{,j}\,dx^j \wedge g_{,k}\,dx^k =: (\nabla f \times \nabla g)\cdot d\mathbf{S} = \mathrm{curl}\,(f\nabla g)\cdot d\mathbf{S} \\ df \wedge dg \wedge dh &= f_{,j}\,dx^j \wedge g_{,k}\,dx^k \wedge h_{,l}\,dx^l =: (\nabla f \cdot \nabla g \times \nabla h)\,d^{\,3}x \end{aligned}$$

Hence write the following expressions in three-dimensional vector notation:

(a) (i) $d^2 f$

(ii) $d(\mathbf{v}\cdot d\mathbf{x})$

(iii) $d(\boldsymbol{\omega}\cdot d\mathbf{S})$

(iv) $d^2(\mathbf{v}\cdot d\mathbf{x})$

(b) (i) $X \lrcorner\, \mathbf{v}\cdot d\mathbf{x}$

(ii) $X \lrcorner\, \boldsymbol{\omega}\cdot d\mathbf{S}$

(iii) $X \lrcorner\, f\,d^{\,3}x$

(c) (i) $d(X \lrcorner\, \mathbf{v}\cdot d\mathbf{x})$

(ii) $d(X \lrcorner\, \boldsymbol{\omega}\cdot d\mathbf{S})$

(iii) $d(X \lrcorner\, f\,d^{\,3}x)$

(d) (i) $\pounds_X f$

(ii) $\pounds_X\,(\mathbf{v}\cdot d\mathbf{x})$

(iii) $\pounds_X(\boldsymbol{\omega}\cdot d\mathbf{S})$

(iv) $\pounds_X(f\,d^{\,3}x)$

(v) $d\pounds_X(\boldsymbol{\omega}\cdot d\mathbf{S})$