Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M3A16/M4A16 Dynamics II

Date: Thursday 2nd June 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The simple harmonic oscillator has Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2,$$

where m and k are constant.

(i) Solve Hamilton's equations in the form $q = q(q_0, p_0, t)$, $p = p(q_0, p_0, t)$, for the evolution of the oscillator in terms of time t and the initial data q_0, p_0 at $t = 0$.

(ii) Show that the Hamiltonian flow preserves area in the (q,p) phase plane.

(iii) Show that the perturbations Δq_0 and Δp_0 in the initial data lead to perturbations Δq and Δp in the solution in (i) which remain bounded in time.

(iv) Explain briefly how (ii) is a general result and why (iii) is not general, even for integrable systems.

2. A system with Hamiltonian H has coordinates q_i, momenta p_i $(i = 1, \ldots, n)$ and Lagrangian

$$L = \sum_{i=1}^{n} p_i \dot{q}_i - H.$$

A canonical transformation is effected to new coordinates Q_i and momenta P_i $(i = 1, \ldots, n)$ using an F_2 generating function, so that the new Lagrangian is

$$L - \frac{d}{dt}\left[F_2(q_i, P_i, t) - \sum_{i=1}^{n} Q_i P_i\right].$$

(i) Show that $Q_i = \dfrac{\partial F_2}{\partial P_i}$ and $p_i = \dfrac{\partial F_2}{\partial q_i}$ $(i = 1, \ldots, n)$ and find the new Hamiltonian K in terms of H and F_2.

(ii) Show that $F_2(q, P, t) = -\frac{1}{2}\omega q^2 \cot P$, with ω constant, generates the canonical transformation $Q = \dfrac{1}{2\omega}(p^2 + \omega^2 q^2)$, $P = -\tan^{-1}\left(\dfrac{\omega q}{p}\right)$.

(iii) Find the canonical transformation $(Q, P) \longmapsto (\overline{Q}, \overline{P})$ which is generated by $\overline{F}_2(Q, \overline{P}, t) = (Q - \Omega t)\overline{P}$, where Ω is constant.

(iv) Using the results from (ii) and (iii) find the generating function $\widetilde{F}_2(q, \overline{P}, t)$ which effects the canonical transformation $(q,p) \longmapsto (\overline{Q}, \overline{P})$.

3. Hamilton's equations with Hamiltonian $H(q_i, p_i)$ $(i = 1, \ldots, n)$ may be expressed in the symplectic form $\dot{\underline{\eta}} = J\dfrac{\partial H}{\partial \underline{\eta}}$, where $\underline{\eta}$ is defined suitably and $J = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$.

(i) Show that the $2n \times 2n$ Jacobian matrix M of a canonical transformation $(q_i, p_i \longmapsto (Q_j, P_j)$ $(i, j = 1, \ldots, n)$ satisfies the symplectic condition $MJ\widetilde{M} = J$, where $\widetilde{M}$ is the transpose of M.

(ii) Using (i) show that the inverse transformation $(Q_i, P_i) \longmapsto (q_j, p_j)$ is also canonical.

(iii) Using (i) or (ii) find constants α and β such that the transformation

$$Q = q^{\alpha}\ \cos(\beta p), \qquad\qquad P = q^{\alpha} \sin(\beta p)$$

is canonical.

(iv) When the transformation in (iii) is canonical evaluate the Poisson brackets $[u, v]$ in the cases

(a) $u = Q, \quad v = P,$

(b) $u = P^2 + Q^2, \quad v = P/Q.$

4. A point projectile of mass m moves in the vertical (q_1, q_2) plane near the surface of the Earth $(q_2 = 0)$, so that q_1, q_2, p_1, p_2 are Cartesian coordinates and momenta.

(i) Show that the Hamilton-Jacobi equation for Hamilton's Principal Function S is

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q_1}\right)^2 + \left(\frac{\partial S}{\partial q_2}\right)^2\right] + mg\, q_2 + \frac{\partial S}{\partial t} = 0.$$

(ii) By writing $S \equiv W(q_1,\, q_2,\, \alpha_1,\, \alpha_2) - \alpha_1 t$ and exploiting the cyclic coordinate, find S to generate a canonical transformation to new coordinates $Q_1 = \beta_1$ and $Q_2 = \beta_2$ and momenta $P_1 = \alpha_1$ and $P_2 = \alpha_2$, where α_i and β_i $(i = 1, 2)$ are constants.

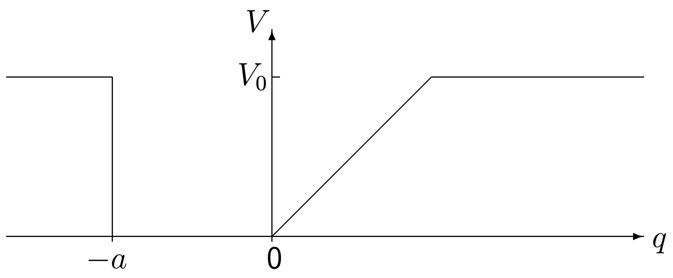
(iii) Hence solve for $q_1(t)$, $q_2(t)$, $p_1(t)$, $p_2(t)$ after expressing α_i and β_i in terms of the initial data $q_{i0} \equiv q_i(0)$, $p_{i0} \equiv p_i(0)$ $(i = 1, 2)$.

(iv) State clearly the physical significance of the constants α_1 and α_2.

5. A particle of mass m moves along the q axis in the potential well

$$V(q) = \begin{cases} V_0 & \text{for} \quad |q| \geq a, \\ 0 & \text{for} \quad -a < q < 0, \\ V_0 \dfrac{q}{a} & \text{for} \quad 0 \leq q < a, \end{cases}$$

where m, V_0 and a are positive constants.



(i) Sketch the phase portrait for this system.

(ii) Show that the action I for the libration motions is given in terms of energy E by

$$I = \frac{a}{\pi}(2mE)^{1/2}\left(1 + \frac{2}{3}\frac{E}{V_0}\right),$$

and find the period τ in terms of E.

(iii) State the principle of adiabatic invariance and use it to show that, if V_0 and m remain constant, but a decreases slowly, then there is a critical value of $a = \overline{a}$ when the particle must leave the potential well. Find how $\overline{a}$ depends on the initial energy E_0 of the particle when $a = a_0$.