Imperial College London

UNIVERSITY OF LONDON

Course:	M3A16
Setter:	Berkshire
Checker:	Gibbons
Editor:	Wu
External:	Vivaldi
Date:	March 9, 2005

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M3A16 Dynamics II

Date: Friday, 28th May   Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

1. Show that the equation of motion of a damped harmonic oscillator

$$m\ddot{q} + 2\mu\dot{q} + kq = 0 \qquad (\text{with } m, k, \mu > 0)$$

can be derived from a Lagrangian

$$L = e^{\frac{2\mu t}{m}} \left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right].$$

Construct the Hamiltonian $H(q, p, t)$ and write down Hamilton's equations.

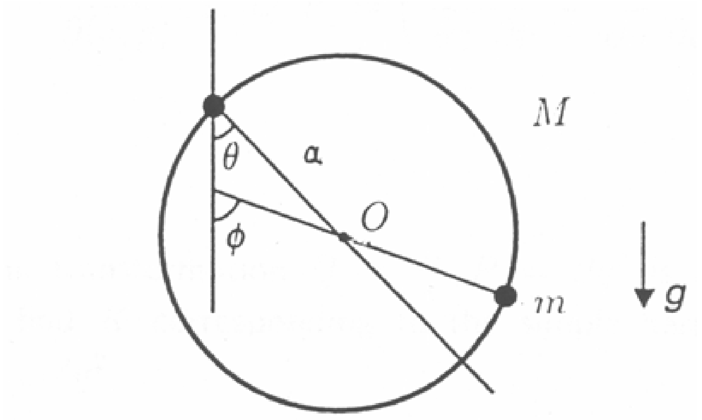
Show that the Hamiltonian

$$K(Q, P) = \frac{P^2}{2m} + \frac{kQ^2}{2} + \frac{\mu}{m}QP$$

leads to equations representing the motion of the same damped harmonic oscillator if $Q = qe^{\frac{\mu t}{m}}$ and express P as a function of q, p, t.

In the case of light damping $\left[\mu < (km)^{1/2}\right]$ show that, by a rotation of coordinates $(Q, P) \to (\overline{Q}, \overline{P})$ in the Q, P phase space, then $K = \lambda_1\overline{Q}^2 + \lambda_2\overline{P}^2$ and the trajectories are ellipses.

2. One point of a uniform circular hoop of mass M, centre O and radius a is fixed and the hoop is free to move in a vertical plane through the fixed point. A bead of mass m slides along the hoop and there is no friction



A suitable Lagrangian is

$$L = Ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\left[\dot{\theta} + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right] + Mga\cos\theta + mga(\cos\theta + \cos\phi)$$

where θ, ϕ are the angles between the vertical and diameters of the hoop through the fixed point, the bead respectively.

(i) Write down Lagrange's equations and, by linearising these, show that the normal frequencies of smalll vibration about the position of stable equilibrium are

$$\left(\frac{g}{2a}\right)^{1/2} \quad \text{and} \quad \left[\left(\frac{M+m}{M}\right)\frac{g}{a}\right]^{1/2}.$$

Describe the physical behaviour of the system in each normal mode.

(ii) Find the generalised momenta p_θ, p_ϕ and show how the Hamiltonian corresponding to L may be constructed. State how many symmetries and independent constants of the motion you would expect for the *general* motion of this system, i.e, for θ, ϕ *not* small. What is the implication of this?

3. If $H(q,p)$ is the Hamiltonian of a dynamical system and a transformation $Q \equiv Q(q,p), P \equiv P(q,p)$ yields $H(q,p) \equiv K(Q,P)$, show explicitly that the transformation is canonical, provided that

$$\frac{\partial(Q,P)}{\partial(q,p)} \equiv [Q,P] = \left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) = 1.$$

Then:

(i) Show that the transformation $Q = \frac{\alpha p}{q}, P = \beta q^2$ is canonical iff $\beta = -\frac{1}{2\alpha}$ and, in this case, find K corresponding to the simple harmonic oscillator Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}mw^2q^2$.

(ii) Show that the Hamilton equation for $Q(t)$ may then be written

$$\dot{Q} = -\frac{Q^2}{m\alpha} - m\alpha w^2.$$

Find the Hamilton equation for $P(t)$.

(iii) Solve the Hamilton equations in (ii) for $Q(t), P(t)$.

(iv) Hence find the solution for $q(t), p(t)$ for the simple harmonic oscillator when the initial data is $q(0) = q_0, p(0) = p_0$.

4. Hamilton's Principal Function $S(q, P, t)$ is the generating function $[' F_2' \, type]$ of a canonical transformation to constant coordinate $Q = \beta$ and momentum $P = \alpha$, satisfying

$$p = \frac{\partial S}{\partial q}, \quad Q = \frac{\partial S}{\partial P}$$

and the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0.$$

A particle of mass m moves along the $\frac{1}{2}$-line $q > 0$ so that its Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{\mu m}{q} + \frac{1}{2}\frac{\lambda}{q^2},$$

with λ, μ positive constants.

Show that Hamilton's Principal Function can have the form

$$S = (2m)^{1/2} \int \left(\alpha + \frac{\mu m}{q} - \frac{1}{2}\frac{\lambda}{q^2}\right)^{1/2} dq - \alpha t$$

where α is the energy.

Hence find β and show that the period τ of oscillations between q_1 and q_2 is

$$\frac{\pi \mu}{2^{1/2}} \left(\frac{m}{-\alpha}\right)^{3/2}$$

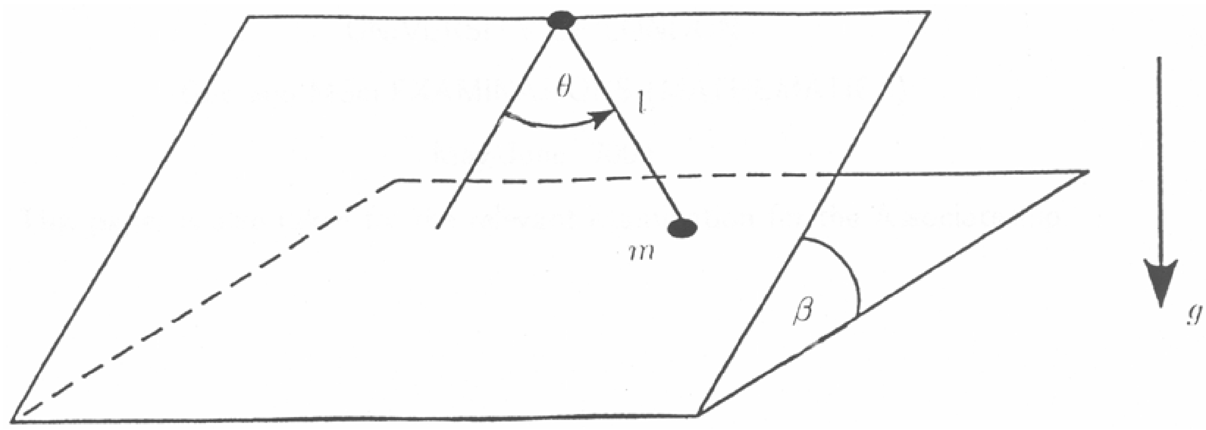
where $q_1 q_2 = -\frac{\lambda}{2\alpha}$ and $q_1 + q_2 = -\frac{\mu m}{\alpha}$.

[The integral

$$\int_{q_1}^{q_2} \frac{q dq}{[(q_2 - q)(q - q_1)]^{1/2}} (0 < q_1 < q_2)$$

is evaluated easily using e.g. $q = q_1 \cos^2 \phi + q_2 \sin^2 \phi$.]

5. A simple pendulum of length l, performs oscillations on a smooth inclined plane tilted at a fixed angle β to the horizontal



(i) Show that a suitable Hamiltonian is

$$H = \frac{p^2}{2ml^2} + mgl \ \sin\beta(1 - \cos q) \equiv \text{Energy } E,$$

where $q \equiv \theta, p = ml^2\dot{\theta}$.

When the oscillations are small, so that $\cos q \simeq 1 - \frac{1}{2}q^2$, show that each phase plane trajectory is an ellipse.

(ii) Hence or otherwise show that the action I is given by

$$I = \left(\frac{l}{g\sin\beta}\right)^{\frac{1}{2}} E$$

and find the dependence of q on the angle variable

$$\phi \equiv \frac{\partial}{\partial I}\int^{q} p \ dq.$$

Hence express q and p as functions of time t.

(iii) If the inclination angle β of the plane is changed very slowly compared to the period of the pendulum, show, using the principle of adiabatic invariance, that the amplitude of angular oscillation of the pendulum varies as $(\sin\beta)^{-\frac{1}{4}}$ and find how its maximum speed varies.