Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

# M3A13 / M4A13

# Waves

Date: Monday, 21st May 2007 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

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1. Small displacements $u(x,t)$ of an elastic beam are described by the Lagrangian

$$\mathcal{L} = \int \left(\tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_{xx}^2\right)\, dx.$$

(a) Determine the Euler–Lagrange equation for the action that corresponds to a Lagrangian density of the form $L(x,t,u,u_t,u_x,u_{xx})$, and hence show that the beam obeys the evolution equation

$$u_{tt} + u_{xxxx} = 0.$$

(b) By considering the chain rule for partial derivatives in the form

$$\left.\frac{\partial L}{\partial t}\right|_x = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial u}u_t + \frac{\partial L}{\partial u_t}u_{tt} + \frac{\partial L}{\partial u_x}u_{xt} + \frac{\partial L}{\partial u_{xx}}u_{xxt},$$

and using the Euler–Lagrange equation, or otherwise, derive an energy equation for the beam in the form

$$\partial_t E + \partial_x F = 0,$$

where $E$ is the energy density and $F$ is the energy flux. Comment briefly on the physical interpretation of the terms in $E$.

(c) For a plane wave $u = \exp[i(kx - \omega t)]$, calculate the averages of $E$ and $F$ over a wave period. Verify that the average energy propagates with the group velocity.

2. The displacement $\eta(x,t)$ of a stretched string with line density $\sigma$ and tension $\tau$ obeys

$$\sigma\eta_{tt} = \tau\eta_{xx}.$$

(a) Suppose that a particle of mass M is attached to the string at $x = 0$. Show that the particle's equation of motion is

$$M\eta_{tt}\big|_{x=0} = \tau\left[\eta_x\right]_{x=0-}^{x=0+}.$$

(b) Suppose that a wave of the form $\exp[i(kx - \omega t)]$ is generated at large, negative $x$. Find the complex amplitudes $R$ and $T$ of the reflected and transmitted waves.
Verify that $|R|^2 + |T|^2 = 1$. What does this mean physically?

(c) Comment briefly on the behaviour of $R$ and $T$ when $M$ is large or small, and write down the relevant dimensionless combination of parameters.

3. The motion of an incompressible fluid of unit density in a frame rotating with angular velocity $\mathbf{\Omega}$ is given by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) = -\nabla p,$$

where $\mathbf{u}$ is the fluid velocity, and $p$ the pressure.

(a) Show that linear waves in a fluid rotating about the $z$ axis obey the equation

$$\nabla^2 P_{tt} + 4\Omega^2 P_{zz} = 0,$$

for some modified pressure $P$ to be determined.

(b) Hence derive the dispersion relation for these waves in the form

$$\omega = \pm 2\Omega \cos\theta,$$

where $\theta$ is the angle between the wave vector and the rotation axis.

Find the angle between the phase velocity $\mathbf{c}_{\mathrm{p}}$ and the group velocity $\mathbf{c}_{\mathrm{g}}$.

(c) A small sphere within the fluid oscillates with frequency $\omega > 2\Omega$.

Find an expression for the disturbance generated in the fluid outside the sphere.

Comment on the shape of the disturbance in the limits

(i) $\omega \to \infty$,

(ii) $\omega \to 2\Omega$ from above.

Is this disturbance an evanescent wave?

[Hint: $\phi = 1/r$ is a solution of Laplace's equation in three dimensions when $r \neq 0$.]

4. Solutions to a scalar wave equation are given approximately by

$$u(\mathbf{x},t) = A(\mathbf{x},t)e^{i\theta(\mathbf{x},t)/\epsilon},$$

where $0 < \epsilon \ll 1$. The local frequency and wave vector defined by

$$\omega = -\frac{1}{\epsilon}\frac{\partial\theta}{\partial t}, \quad \mathbf{k} = \frac{1}{\epsilon}\frac{\partial\theta}{\partial\mathbf{x}}$$

obey the dispersion relation

$$\omega = \Omega(\mathbf{k};\mathbf{x},t).$$

(a) Derive the ray-tracing equations

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial\Omega}{\partial\mathbf{x}} \quad \text{along the rays} \quad \frac{d\mathbf{x}}{dt} = \frac{\partial\Omega}{\partial\mathbf{k}}.$$

(b) The dispersion relation for sound waves in a stratified medium is given by $\omega = \omega_0|\mathbf{k}|z$, where $\omega_0$ is a positive constant. Consider rays moving in the $xz$ plane.

(i) Show that the ray paths form arcs of circles, and give an expression for their radii.

(ii) Show that rays moving towards the line $z = 0$ do not reach the line in finite time.

[Hint for (b): Begin by finding any conserved quantities, for which standard results may be quoted without proof.

$$\frac{d}{dz}\cosh^{-1}(a/z) = -\frac{a}{z}\left(a^2 - z^2\right)^{-1/2}.$$ ]

5. (a) The intrinsic frequency of waves on deep water is given by

$$\widehat{\omega}^2 = g|\mathbf{k}|.$$

Using the transformation $\mathbf{x}_{\mathrm{old}} = \mathbf{x}_{\mathrm{new}} - Ut\,\widehat{\mathbf{x}}$, show that the dispersion relation as seen from a frame moving with constant velocity $U$ in the *negative* $x$ direction may be written as

$$\omega = k \pm (k^2 + l^2)^{1/4},$$

after introducing suitable dimensionless variables.

(b) A steady wave pattern is generated by a point source located at $\mathbf{x} = 0$ in the moving frame. Given that the disturbance may be written as

$$\eta(x, y) = \int_{-\infty}^{\infty} F(k) e^{i(kx+ly)} dk,$$

use the method of stationary phase to show that the far-field disturbance is confined to a cone, and find the angle of that cone.