1. (a) Using the substitution $\phi = \psi_x$, derive the linearised Korteveg–de Vries equation

$$\phi_t + \alpha\phi_x + \beta\phi_{xxx} = 0$$

from the Lagrangian $\mathcal{L} = \int L\,dx$, with Lagrangian density

$$L = \frac{1}{2}\psi_t\psi_x + \frac{1}{2}\alpha\psi_x^2 - \frac{1}{2}\beta\psi_{xx}^2.$$

(b) Consider the chain rule for partial derivatives in the form

$$\frac{\partial L}{\partial t}\Big|_x = \frac{\partial L}{\partial t}\Big|_{x,\psi,\psi_x,\psi_t,\psi_{xx}} + \frac{\partial L}{\partial \psi}\psi_t + \frac{\partial L}{\partial \psi_t}\psi_{tt} + \frac{\partial L}{\partial \psi_x}\psi_{xt} + \frac{\partial L}{\partial \psi_{xx}}\psi_{xxt}.$$

On the left L is treated as a function of x and t only, and on the right L is treated as a function of x, t, ψ and its derivatives.

By using this relation and the Euler–Lagrange equation, show that the quantity E given by

$$E = \frac{1}{2}\beta\psi_{xx}^2 - \frac{1}{2}\alpha\psi_x^2,$$

where $\psi(x,t)$ is a solution of the equation derived from the above Lagrangian, satisfies a conservation law of the form

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

and give an expression for the quantity F.

Hint: For part (b) you may find it useful to express E and F in terms of a general Lagrangian $L(\psi, \psi_t, \psi_x, \psi_{xx})$ first.

2. (a) Suppose that solutions to a scalar wave equation are given approximately by

$$u(\mathbf{x},t) = A(\mathbf{x},t)e^{i\theta(\mathbf{x},t)/\epsilon}.$$

The parameter ϵ is sufficiently small that the dispersion relation

$$\omega = \Omega(\mathbf{k};\mathbf{x},t)$$

holds for the local frequency and wave vector defined by

$$\omega = -\frac{1}{\epsilon}\frac{\partial\theta}{\partial t}, \quad \mathbf{k} = \frac{1}{\epsilon}\frac{\partial\theta}{\partial\mathbf{x}}.$$

Hence derive the ray-tracing equations

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial\Omega}{\partial\mathbf{x}} \quad \text{along rays} \quad \frac{d\mathbf{x}}{dt} = \frac{\partial\Omega}{\partial\mathbf{k}}.$$

(b) Consider the two-dimensional wave equation

$$\frac{\partial^2\phi}{\partial t^2} - c(z)^2\nabla^2\phi = 0$$

for the scalar variable $\phi(x,z,t)$.

Suppose the wave speed c is given by

$$c(z) = \frac{c_+ + c_-}{2} + \frac{c_+ - c_-}{2}\tanh z,$$

which varies smoothly and monotonically from the value c_- as $z \to -\infty$ to some other value c_+ as $z \to +\infty$.

Plane waves with frequency ω and wave vector $\mathbf{k} = |\mathbf{k}|(\cos\theta_-, \sin\theta_-)$ are generated at large, negative z and propagate in the positive z direction through the region of varying c.

Use the conservation properties of the ray-tracing equations to find an ordinary differential equation for the ray paths.

Without attempting to solve this ordinary differential equation, sketch a typical ray path, and deduce Snell's law,

$$\frac{\cos\theta_+}{c_+} = \frac{\cos\theta_-}{c_-},$$

for the angle θ_+ that the emerging ray paths make to the horizontal as $z \to +\infty$.

(c) Snell's law also holds when one considers a sharp transition at $z = 0$ between two different media, each with uniform wave speed $c_\pm$. Why is this?

3. (a) An acoustic wave guide consists of a rectangular channel with rigid walls at $x = 0$ and $x = a$, and at $y = 0$ and $y = b$. It contains air with mean density ρ_0. By seeking separable solutions of the form

$$\phi(x, y, z, t) = X(x)Y(y)e^{i(kz-\omega t)}$$

to the three dimensional wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0,$$

with constant sound speed c, show that the frequency ω satisfies the dispersion relation

$$\omega^2 = c^2 \left[k^2 + \frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2} \right],$$

where n and m are integers.

(b) Suppose that a wave maker generates waves for one value of n and m at the end of the wave guide, with frequency $\omega > c f_{nm}$, where

$$f_{nm} = \left[\frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2} \right]^{1/2}.$$

Write down the energy density and energy flux for sound waves. By integrating these quantities across an xy cross section of the wave guide, and averaging over a wave period, show that the average integrated energy density travels down the wave guide with the group velocity obtained from the dispersion relation above.

Why does the energy not leak through the sides of the wave guide?

(c) Briefly describe what would happen if the wave maker oscillated with a lower frequency $\omega < c f_{nm}$.

Hint: In sound waves the pressure perturbation is related to the velocity potential by $p' = -\rho_0 \dfrac{\partial \phi}{\partial t}$.

4. (a) The pressure p_0 and density ρ_0 in an isothermal atmosphere at rest under hydrostatic balance are given by

$$\rho_0(z) = \rho_0(0)\exp(-z/H), \quad p_0(z) = p_0(0)\exp(-z/H),$$

where the constant H is called the scale height, and $p_0 = RT_0\rho_0$.

Small perturbations around this rest state,

$$\rho = \rho_0(z) + \rho'(x,y,z,t), \quad p = p_0(z) + p'(x,y,z,t), \quad \mathbf{u}' = \mathbf{u} = (u,v,w),$$

are governed by the anelastic equations

$$\partial_t \mathbf{u}_\perp + \nabla_\perp(p'/\rho_0) = 0, \quad \partial_t w - g\left(\frac{\theta'}{\theta_0}\right) + \frac{\partial}{\partial z}\left(\frac{p'}{\rho_0}\right) = 0,$$
$$\nabla\cdot(\rho_0\mathbf{u}) = 0, \quad \frac{\partial}{\partial t}\left(\frac{\theta'}{\theta_0}\right) + \frac{N^2}{g}w = 0,$$

where $\nabla_\perp = (\partial_x, \partial_y)$, and $\mathbf{u}_\perp = (u,v)$ is the horizontal velocity. The quantity θ' is related to the pressure and density fluctuations by

$$\frac{\theta'}{\theta_0} - \frac{1}{c^2}\frac{p'}{p_0} + \frac{\rho'}{\rho_0} = 0.$$

By combining pairs of the above four equations, derive the relations

$$\left(\frac{\partial^2}{\partial t^2} + N^2\right)w + \frac{\partial^2}{\partial z\partial t}\left(\frac{p'}{\rho_0}\right) = 0, \quad \frac{\partial}{\partial t}\left(\frac{1}{H} - \frac{\partial}{\partial z}\right)w + \nabla_\perp^2\left(\frac{p'}{\rho_0}\right) = 0.$$

(b) Show that solutions in which w and p'/ρ_0 are both constant multiples of $\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$ exist provided ω satisfies the dispersion relation

$$\omega^2 = N^2\,\frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2 + ik_z/H}.$$

Hence show that purely oscillatory solutions, for which the frequency ω is purely real, are only possible when k_z is complex. In other words, the amplitude of the oscillations must vary exponentially in z, at a rate to be determined.

Write down the dispersion relation for these purely oscillatory solutions.

(c) Without making detailed calculations, is the group velocity perpendicular to the phase velocity for these purely oscillatory solutions? What happens when the wavelength $2\pi/|\mathbf{k}|$ is very small compared with the scale height H?

5. (a) The dispersion relation for water waves on a layer of finite depth H is given by

$$\omega^2 = gk\tanh(kH),$$

where k is the wavenumber and g is gravity. Solutions to this dispersion relation may be written as

$$\omega = \pm\Omega(k).$$

Sketch $\Omega(k)$, and find the maximum values of the phase and group velocities as functions of wave number.

(b) Suppose the free surface $\eta(x,t)$ is initially at rest at $t=0$, with $\eta(x,0)=\eta_0(x)$ for some function $\eta_0(x)$ that decays rapidly as $x \to \pm\infty$. Show that the subsequent solution may be written as

$$\eta(x,t) = \int_{-\infty}^{\infty} \left[F(k)e^{ikx-\Omega(k)t} + F(k)e^{ikx+\Omega(k)t}\right] dk,$$

for some function $F(k)$ to be written down.

(c) Use the idea behind the method of stationary phase to simplify the above integral at large t, with x/t fixed. Hence explain why the free surface near the rightmost wavefront behaves as though it were evolving under the equation

$$\eta_t + c\eta_x + \gamma\eta_{xxx} = 0,$$

with parameters c and γ to be determined. What are the relevant initial conditions?

(d) Without attempting to calculate the solution explicitly, determine how the maximum displacement near the rightmost wavefront decays with time for large times.

Hint: $\tanh z = z - \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$ for small z.