1. A viscous incompressible fluid of density ρ, kinematic viscosity ν flows in the two-dimensional channel defined by $0 \leqslant y \leqslant hF(x/L)$. If U_0 is a typical flow velocity, write down the conditions which must be satisfied if lubrication theory is to be used to describe the flow.

Show that if $X = x/L$, $Y = y/h$ the dimensionless form of the lubrication equations are

$$U_X + V_Y = 0,$$

$$0 = -P_X + U_{YY},$$

$$0 = -P_Y,$$

where U, V, and P are dimensionless velocities and pressures.

If the pressure difference between X=0 and X=h is $\dfrac{\mu U_0 L}{h^2}$, where $\nu = \dfrac{\mu}{\rho}$, show that P is given by

$$P = \frac{\int_0^X dX/F^3}{\int_0^1 dX/F^3}$$

Show that the dimensionless shear stress acting on the lower wall has a maximum magnitude of A/2 for the case

$$F = 1 + \frac{(X - \frac{1}{2})^2}{2},$$

where

$$A = \int_0^1 \frac{dX}{F^3}.$$

2. A fluid of viscosity μ fills a volume V confined within the surface S and the motion is slow enough for the equations of motion to reduce to the Stokes equations

$$\nabla . \underline{u} = 0,$$

$$\nabla p = \mu \nabla^2 \underline{u}.$$

Show that there is no more than one solution of the above system satisfying

$$\underline{u} = \underline{U^*}$$

on the surface S.

Consider the function

$$\Phi = 2\mu \int_V e_{ij} e_{ij} dV,$$

where V is the volume included within S. If the velocity field $\underline{u}$ is a solution of

$$\nabla . \underline{u} = 0$$

show that Φ is minimised by $\underline{u}$ satisfying the Stokes equations.

3. Consider the steady two-dimensional flow of a viscous fluid of kinematic viscosity ν. Show that the equations of motion under appropriate approximations can be reduced to the boundary layer equations

$$\Psi_y \Psi_{xy} - \Psi_z \Psi_{yy} = UU_x + 2\Psi_{yyy},$$

where U is the fluid velocity at infinity. Show that these equations support a similarity solution

$$\Psi = k_1 x^m f(\eta), \; \eta = \frac{y}{k_2 x^n},$$

with k_1, k_2, m, n unknown constants provided that m+n=1.

Suppose that the above system is used to describe a jet flow symmetric about y=0 with $u \to 0$, $y \to \pm\infty$. Now write down the x momentum equation and show that if U=0 when $y \to \pm\infty$ then $\int_{-\infty}^{\infty} u^2 \, dy$ = constant and deduce that $2m$-n=0.

Hence show that if $k_1 k_2 = 6\nu$ a similarity solution of the above form exists if

$$f''' + 2ff'' + 2f'^2 = 0,$$

Deduce that

$$f' + f^2 = \beta^2$$

where β is constant. Integrate once more to show that $f = \beta \tanh \beta\eta$.

4. Consider the flow of an incompressible fluid of kinematic viscosity ν between cylinders of radii R_1 and R_2. The inner cylinder rotates with an angular velocity $\Omega_1 + \Omega_2 \cos \omega t$ and the other cylinder is at rest. The equations of motion in cylindrical polar coordinates are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0,$$

$$\mathcal{L}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} \dfrac{-v^2}{r} \\ \dfrac{uv}{r} \\ 0 \end{pmatrix} = \frac{-1}{\rho}\nabla p + \nu \mathcal{M}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \nu \begin{pmatrix} -\dfrac{u}{r^2} - \dfrac{2}{r^2}\dfrac{\partial v}{\partial \theta} \\ \dfrac{2}{r^2}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r^2} \\ 0 \end{pmatrix},$$

where ρ is the density,

$$\mathcal{L} \equiv u\frac{\partial}{\partial v} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z},$$

$$\mathcal{M} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

Show that the equations have a solution *(0,v(r,t),0)*. Determine *v(r,t)* and show that the pressure P may be written

$$\mathrm{P} = \rho \int^r v^2(r,t)dr.$$

What form do you expect the solution for v to take in the limit $\omega \to 0$? Write down your predicted form for *v* in that limit.

5. Consider the unidirectional flow $\underline{u} = (\overline{u}(y), 0, 0)$ of an incompressible fluid. Taking the equations of motion in the dimensionless form

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = -p_x + \frac{1}{R}(u_{xx} + u_{yy})$$

$$v_t + uv_x + vv_y = -p_y + \frac{1}{R}(v_{xx} + v_{yy})$$

where R is the Reynolds number, show that small perturbations satisfy the Orr-Sommerfield equation

$$(\overline{u} - c)(v_{yy} - \alpha^2 v) - \overline{u}'' v = \frac{1}{i\alpha R}(\frac{d^2}{dy^2} - \alpha^2)^2 V$$

where α and c are the wavenumber in the x direction and wavespeed of the disturbance. Show that if $v = 0$, $ony = y_1$, y_2, then unstable disturbances cannot exist in the infinite Reynolds number limit unless an inflexion point exists in $\overline{u}$(y).