Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MSc EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3A10/M4A10

Viscous Flow

Date: Wednesday 24th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. A viscous incompressible fluid of viscosity μ flows in the channel defined by

$$0 \le z \le h(x,y)\,.$$

Write down the circumstances under which the flow can be described by the lubrication equations:

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2} = 0\,, \qquad -\frac{\partial p}{\partial y} + \mu\frac{\partial^2 v}{\partial z^2} = 0\,,$$

$$\frac{\partial p}{\partial z} = 0\,, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,.$$

Here u, v, w are the velocity components in the x, y, z directions, p is the pressure.

(a) Give an estimate of the order of magnitude of the largest terms neglected in these equations.

(b) The surface $z = h$ moves in the z direction with velocity W. What are the boundary conditions for the flow?

(c) Find the velocity components u and v and deduce from the equation of continuity and boundary conditions that

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right) = 12\mu W.$$

(d) By writing the above equation in the form

$$\mathrm{div}(h^3 \mathrm{grad}\, p) = 12\mu W,$$

show that if the flow depends only on $r = \sqrt{x^2+y^2}$ and by using the polar coordinate forms of grad and div, then the pressure satisfies

$$\frac{dp}{dr} = \frac{6\mu W r}{h^3} + \frac{A}{rh^3}\,,$$

where A is an arbitrary constant.

(e) A sphere of radius a falls towards the plane $z = 0$ with speed $-W$. When the sphere is close to the plane, show that, if h_0 is the minimum distance between the sphere and wall, then h, the height of a point on the sphere above the plane, is approximately $\left(h_0 + \frac{r^2}{2a}\right)$.

Hence show that

$$\frac{dp}{dr} = \frac{6\mu W r}{\left(h_0 + \frac{r^2}{2a}\right)^3}\,.$$

2. An incompressible viscous fluid of kinematic viscosity ν fills the two-dimensional channel defined by $0 \leq y \leq d$.

The flow is at rest until $t = 0$ when a uniform pressure gradient is applied in the x direction and maintained for $t \geq 0$.

Show that a unidirectional flow is possible, and find the flow for $t > 0$.

For small values of $d^2/(\nu t)$ show that

$$u \simeq \frac{Gy(d-y)}{2\mu} - \frac{4Gd^2}{\mu\pi^3}\left\{ \sin\left(\frac{\pi y}{d}\right) e^{-\pi^2 \nu t\, d^{-2}} + O\left(e^{-9\pi^2 \nu t\, d^{-2}}\right)\right\}.$$

3. Consider the flow of a viscous incompressible fluid of kinematic viscosity ν in the half plane $y \geq 0$. Fluid enters the wall $y = 0$ at right angles with constant speed V. Far from the wall the x velocity component tends to the time dependent speed $U_0 \cos \omega t$.

Write down the equations which determine the above flow based on the assumption that the velocity field and pressure gradient in the x direction are independent of x.

Determine (u, v) for the above flow. Discuss the limiting cases :

(a) $V \to \infty$,

(b) $\omega \to 0$,

(c) $\omega \to \infty$.

4. Under what circumstances do the Stokes equations

$$-\nabla p + \mu \nabla^2 \mathbf{u} = 0, \qquad \nabla.\mathbf{u} = 0$$

represent a good approximation to the Navier Stokes equations?

Suppose that a volume V of fluid is bounded by a surface S and that $\mathbf{u}$ is specified on S. Show that the solution of the Stokes equations in V is unique.

Show that for a two-dimensional Stokes flow the stream function $\psi(x, y)$ satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \psi = 0 .$$

5. Give an argument to show how the steady Navier Stokes equations in two dimensions can be reduced to the boundary layer equations

$$uu_x + vu_y = UU'(x) + \nu u_{yy}\,,$$

$$u_x + v_y = 0\,,$$

where $U(x)$ is the inviscid slip velocity.

Show that the appropriate boundary conditions are

$$u = v = 0 \;\text{ on }\; y = 0\,, \quad u \to U(x)\,, \quad y \to \infty\,.$$

Now suppose that $U(x) = kx$. Show that the above boundary layer equations allow a solution with

$$u = kxf'(\zeta), \quad v = -(\nu k)^{\frac{1}{2}}\, f(\zeta), \quad \zeta = y\left(\frac{k}{\nu}\right)^{\frac{1}{2}},$$

where f satisfies

$$f''' + ff'' - f'^2 + 1 = 0\,,$$

$$f(0) = f'(0) = 0\,, \quad f'(\infty) = 1\,.$$

Deduce that the pressure $p(x)$ within the boundary layer is given by

$$-\left(\frac{p - p_0}{\rho}\right) = \tfrac{1}{2}k^2x^2\,,$$

where p_0 is a constant and ρ is the density.

Show that the Navier Stokes equations for the flow allow an exact solution with the same velocity field found above. Show that the pressure is now given by

$$-\left(\frac{p - p_0}{\rho}\right) \;=\; \tfrac{1}{2}k^2x^2 + k\nu f'(\zeta) + \tfrac{1}{2}k\nu f^2\,.$$