Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2S3 Statistical Theory I

Date: Monday, 22nd May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are given on pages 4 and 5.

1. (a) Suppose that you are given independent identically distributed random variables (iid rvs) $X_1, X_2, \ldots, X_n$ in a regular estimation problem for a single unknown parameter θ.

   (i) Define the *total efficient score* $U_\bullet(\theta)$.

   (ii) Define the *total Fisher information* $I_\bullet(\theta)$.

   (b) Let $X_1, X_2, \ldots, X_n$ be iid rvs from $N(0, \theta)$ where $\theta > 0$.

   (i) Find $U_\bullet(\theta)$ and $I_\bullet(\theta)$.

   (ii) Determine the maximum likelihood estimates of θ, $\sqrt{\theta}$ and $\ln \theta$, and their asymptotic variances as $n \to \infty$.

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed *Bernoulli*(θ) random variables having probability mass function

$$f_{X|\theta}(x|\theta) \;=\; \theta^x (1-\theta)^{1-x} \qquad (x = 0,\, 1)$$

where $0 < \theta < 1$.

   (a) Show that X_1 is unbiased for θ.

   (b) Explain, without proof, why $S = \sum_1^n X_i$ is a complete sufficient statistic for θ.

   (c) Explain, without proof, why S/n is a uniformly minimum variance unbiased (UMVU) estimator for θ.

   (d) State, without proof, the Rao-Blackwell Theorem for finding UMVU estimators.

   (e) Demonstrate the application of the Rao-Blackwell Theorem by finding the distribution of X_1 given $S = s$, and showing that its expectation is s/n.

3. (a) Write down the form of the joint probability density function or joint probability mass function $f_{\mathbf{X}|\theta}(\mathbf{x}|\theta)$ for the one-parameter exponential family (EF) parameterised by $\theta \in \Theta \subseteq \mathbb{R}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

   (b) Identify a complete minimal sufficient statistic from the one-parameter EF.

   (c) Identify each of the following as EF or not. *Give your reasoning.*

   (i) *Uniform*$(-\theta,\, \theta)$ where $\theta > 0$

   (ii) $f_{X|\theta}(x|\theta) \;=\; \exp\{-(x-\theta)\} \qquad (x \in (\theta, \infty))$

   (iii) $f_{X|\theta}(x|\theta) \;=\; 2\,\dfrac{x+\theta}{1+2\theta} \qquad (x \in (0,1))$ where $\theta > 0$

   (iv) $f_{X|\theta}(x|\theta) \;=\; \dfrac{\theta^x \ln \theta}{\theta - 1} \qquad (x \in (0,1))$ where $\theta > 0$

   For those that are EF, write $f_{\mathbf{X}|\theta}(\mathbf{x}|\theta)$ in the form (a) for independent identically distributed random variables $X_1, X_2, \ldots, X_n$, and obtain a complete minimal sufficient statistic.

4. (a) What is the *monotone likelihood ratio criterion*?

(b) A test is to be made of the null hypothesis H_0 that a distribution is *Uniform*$(0,1)$. The alternative hypothesis H_1 is that the distribution has the probability density function

$$f_{X|\theta}(x|\theta) \;=\; (\theta+1)\,x^{\theta} \qquad (x \in (0,1))$$

where $\theta > 0$.

(i) Show that the likelihood ratio with data $x_1, x_2, \ldots, x_n$ has monotone likelihood ratio, and hence find the form of the appropriate rejection region for the test.

(ii) By showing that the transformation $Y = -\ln X$ gives *Exponential*(1) random variables under H_0, show that a size α test can be carried out by using values of the cumulative distribution function $F_\Gamma(z)$ of *Gamma*$(n,1)$.

(iii) Give an expression in terms of $F_\Gamma(z)$ for the power $\beta(\theta)$ of the test.

5. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables from the *Delayed Exponential* distribution having the probability density function (pdf)

$$f_{X|\theta}(x \mid \theta) \;=\; \theta\,\exp\{-\theta(x-1)\} \qquad (x > 1)$$

where $\theta > 0$ is unknown.

Suppose that the prior distribution for θ is *Exponential*(λ), where $\lambda > 0$ is known.

Determine

(a) the likelihood function,

(b) the posterior pdf,

(c) the posterior mean,

(d) the posterior variance,

(e) the posterior mode.

Distribution	$f(x \mid \theta)$	$x \in \mathbb{X}$	$\theta \in \Theta$
Bernoulli(θ)	$\theta^x(1-\theta)^{1-x}$	$x = 0, 1$	$0 < \theta < 1$
(Discrete) Uniform(k)	$\frac{1}{k}$	$x = 1, 2, \ldots, k$	
Binomial(n, θ)	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$	$x = 0, 1, \ldots, n$	$0 < \theta < 1$
Poisson(λ)	$\frac{\lambda^x e^{-\lambda}}{x!}$	$x = 0, 1, 2, \ldots$	$\lambda > 0$
Geometric(θ)	$(1-\theta)^{x-1}\theta$	$x = 1, 2, \ldots$	$0 < \theta < 1$
NegativeBinomial(n, θ)	$\binom{x+n-1}{n-1}(1-\theta)^x\theta^n$	$x = 0, 1, 2, \ldots$	$0 < \theta < 1,\ n = 1, 2, \ldots$
Uniform(α, β)	$\frac{1}{\beta - \alpha}$	$\alpha < x < \beta$	$\alpha < \beta$
Exponential(λ)	$\lambda \exp(-\lambda x)$	$x > 0$	$\lambda > 0$
Gamma(ν, λ)	$\frac{1}{\Gamma(\nu)}\lambda(\lambda x)^{\nu-1}\exp(-\lambda x)$	$x > 0$	$\lambda > 0,\ \nu > 0$
Cauchy(α, β)	$\frac{1}{\pi\beta\{1 + (\frac{x-\alpha}{\beta})^2\}}$	$-\infty < x < \infty$	$\beta > 0$
$N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$	$-\infty < x < \infty$	$\sigma^2 > 0$
Beta(α, β)	$\frac{1}{B(\alpha, \beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$\alpha > 0,\ \beta > 0$
Weibull(α, β)	$\beta\alpha x^{\alpha-1}\exp(-\beta x^\alpha)$	$x > 0$	$\alpha > 0,\ \beta > 0$
χ^2_k	$\frac{1}{2^{k/2}\Gamma(k/2)}x^{(k/2)-1}\exp(-\frac{1}{2}x)$	$x > 0$	$k = 1, 2, \ldots$
t_m	$\frac{\Gamma((m+1)/2)}{\Gamma(m/2)\sqrt{\pi m}}\ (1 + \frac{x^2}{m})^{-(m+1)/2}$	$-\infty < x < \infty$	$m = 1, 2, \ldots$
Pareto(θ)	$\frac{\theta}{x^{\theta+1}}$	$x > 1$	$\theta > 0$

Distribution	$E(X)$	$\operatorname{var}(X)$	$G_X(z)$ or $M_X(t)$
Bernoulli(θ)	θ	$\theta(1-\theta)$	$1-\theta+\theta z$
(Discrete) Uniform(k)	$(k+1)/2$	$(k^2-1)/12$	$z(1-z^k)/\{k(1-z)\}$
Binomial(n,θ)	$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta z)^n$
Poisson(λ)	λ	λ	$\exp\{-\lambda(1-z)\}$
Geometric(θ)	$\dfrac{1}{\theta}$	$\dfrac{1-\theta}{\theta^2}$	$\dfrac{\theta z}{1-z(1-\theta)}$
NegativeBinomial(n,θ)	$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta z}{1-z(1-\theta)}\right)^n$
Uniform(α,β)	$\frac{1}{2}(\alpha+\beta)$	$\frac{1}{12}(\beta-\alpha)^2$	$(e^{\beta t}-e^{\alpha t})/\{(\beta-\alpha)t\}$
Exponential(λ)	$1/\lambda$	$1/\lambda^2$	$\lambda/(\lambda-t)$
Gamma(ν,λ)	ν/λ	ν/λ^2	$\{\lambda/(\lambda-t)\}^\nu$
Cauchy	none	none	none
$N(\mu,\sigma^2)$	μ	σ^2	$\exp(\mu t+\frac{1}{2}\sigma^2t^2)$
Beta(α,β)	$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	${}_1F_1(\alpha;\beta;t)$
Weibull(α,β)	$\beta^{-1/\alpha}\Gamma\left(1+\dfrac{1}{\alpha}\right)$	$\beta^{-2/\alpha}\left\{\Gamma\left(1+\dfrac{2}{\alpha}\right) - \left[\Gamma\left(1+\dfrac{1}{\alpha}\right)\right]^2\right\}$	none
χ^2_k	k	$2k$	$(1-2t)^{-k/2}$
t_m	0	$\dfrac{m}{m-2}$	none
Pareto(θ)	$\dfrac{\theta}{\theta-1}$	$\dfrac{\theta}{(\theta-1)^2(\theta-2)}$	none