Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M2S3 Statistical Theory I

Date: Tuesday, 24th May 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are given on pages 5 and 6.

1. Consider the one-parameter exponential family of distributions having the probability density function (pdf)

$$f_{X|\theta}(x \mid \theta) \;=\; \exp\{\, \frac{1}{x}\,\phi(\theta) + \alpha(x) + \beta(\theta)\}\,.$$

(a) Show that, for a random sample $x_1, x_2, \ldots, x_n$ from the distribution, their harmonic mean, $t = \left(\frac{1}{n}\sum_1^n \frac{1}{x_i}\right)^{-1}$, is a sufficient statistic for θ.

(b) Show that if t is the maximum likelihood estimate (MLE) of θ then

$$\phi'(\theta) = -\theta\beta'(\theta)\,. \qquad (*)$$

(c) If $(*)$ can be assumed, find a function $\psi(\theta)$ that has an unbiased estimator $\widehat{\psi}$ for which $\mathrm{var}\,(\widehat{\psi})$ satisfies the Cramér-Rao lower bound.

(d) Show that the pdf

$$f_{X|\theta}(x \mid \theta) \;=\; \frac{\theta}{x^2}\,\exp\left(-\frac{\theta}{x}\right) \qquad (0 < x < \infty)$$

is a special case of the above, with $(*)$ satisfied.

Write down $\psi(\theta)$ and its unbiased estimator $\widehat{\psi}$, and show that $\mathrm{var}\,(\widehat{\psi}) = \dfrac{1}{n\,\theta^2}$.

Find the asymptotic variance of the MLE of θ.

2. Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed $N(\theta, 1)$.

(a) Show that $\overline{X}$ is a sufficient statistic for θ.

(b) Explain, without proof, why we may take $\overline{X}$ to be complete for θ.

(c) Show that $X_1^2 - 1$ is unbiased for θ^2.

(d) For the case $n = 2$,

(i) show that $X_1 - \overline{X}$ is independent of $\overline{X}$.

(ii) Hence find the distribution of X_1 given $\overline{X}$.

(iii) Find $E(X_1^2 \mid \overline{X})$ and show that $\overline{X}^2 - \frac{1}{2}$ is the minimum variance unbiased estimator (MVUE) of θ^2.

(e) If you can assume that $X_1 - \overline{X}$ is independent of $\overline{X}$ for any n, find the MVUE of θ^2 in the general case of a sample of size n.

3. (a) (i) What is meant by the term *sufficient statistic*?

(ii) What is a *complete statistic*?

(iii) State a sufficient condition for a function of a sufficient statistic t to be the unique unbiased estimator of an unknown parameter θ.

(b) Monthly counts of accidents reported to a safety officer may be modelled as independent *Poisson* (θ) random variables. The random variables are observed on successive months to take values $x_1, x_2, \ldots, x_n$.

(i) Show that $s = \sum_1^n x_i$ is a sufficient statistic for θ .

(ii) The safety officer wishes to estimate the odds in favour of one or more accidents being reported in any future month, that is to say, an estimate of

$$\psi \;=\; \frac{P(X \neq 0)}{P(X = 0)} \,.$$

Find $\psi(\theta)$.
By writing

$$E(t(S)) \;=\; \sum_{s=0}^{\infty} t(s)\, p_S(s)\,,$$

where $p_S(s)$ is the probability mass function for S, find coefficients $t(s)$ which make $t(S)$ an unbiased estimator of $\psi(\theta)$.
Hence find the MVUE for ψ, stating the condition on S that you need to be satisfied, and why you can assume that it is satisfied.

4. (a) State the Neyman-Pearson Lemma.

(b) What is the Monotone Likelihood Ratio Criterion for obtaining a uniformly most powerful (UMP) test of $H_0 : \theta \leq \theta^*$ against $H_1 : \theta > \theta^*$, where θ^* is given?

(c) Find the size α UMP test of $H_0 : \theta \leq \theta^*$ against $H_1 : \theta > \theta^* > 0$, when $X_1, X_2, \ldots, X_n$ are independent and identically distributed *Gamma*$(2, \theta)$ $(\theta > 0)$, ie

$$f_{X|\theta}(x \mid \theta) \;=\; \theta^2 x\, e^{-\theta x} \quad (x > 0)\,.$$

Show how the power function for the test might be found.

5. (a) What is meant by saying that a parameter Θ has an *improper prior distribution*?

(b) Suppose that $X_1, X_2, \ldots, X_n$ given $\Theta = \theta$ are independent and identically distributed *Pareto* (θ), *ie*

$$f_{X|\theta}(x \mid \theta) \;=\; \frac{\theta}{x^{\theta+1}} \qquad (x > 1) \;\text{ where }\; \theta > 0.$$

(i) Show that the geometric mean, $t = \left(\prod_1^n x_i\right)^{\frac{1}{n}}$, is a sufficient statistic for θ.

(ii) Find the maximum likelihood estimator $\widehat{\theta}$ for θ.

(iii) Suppose now that Θ has the improper prior probability density function (pdf) θ^{-1} on $(0, \infty)$. Find the posterior pdf and show that it can be written as the pdf of $c\Psi$, where Ψ has a χ^2_m distribution, with c being a function of the sufficient statistic t. Find c and m.
Hence find the expectation and variance of the posterior distribution of Θ.

Distribution	$f(x \mid \theta)$	$x \in \mathbb{X}$	$\theta \in \Theta$
Bernoulli(θ)	$\theta^x(1-\theta)^{1-x}$	$x = 0, 1$	$0 < \theta < 1$
(Discrete) Uniform(k)	$\frac{1}{k}$	$x = 1, 2, \ldots, k$	
Binomial(n, θ)	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$	$x = 0, 1, \ldots, n$	$0 < \theta < 1$
Poisson(λ)	$\frac{\lambda^x\, e^{-\lambda}}{x!}$	$x = 0, 1, 2, \ldots$	$\lambda > 0$
Geometric(θ)	$(1-\theta)^{x-1}\theta$	$x = 1, 2, \ldots$	$0 < \theta < 1$
NegativeBinomial(n, θ)	$\binom{x+n-1}{n-1}(1-\theta)^x\theta^n$	$x = 0, 1, 2, \ldots$	$0 < \theta < 1,\ n = 1, 2, \ldots$
Uniform(α, β)	$\frac{1}{\beta - \alpha}$	$\alpha < x < \beta$	$\alpha < \beta$
Exponential(λ)	$\lambda \exp(-\lambda x)$	$x > 0$	$\lambda > 0$
Gamma(ν, λ)	$\frac{1}{\Gamma(\nu)}\lambda(\lambda x)^{\nu-1}\exp(-\lambda x)$	$x > 0$	$\lambda > 0,\ \nu > 0$
Cauchy(α, β)	$\frac{1}{\pi\beta\,\{1 + (\frac{x-\alpha}{\beta})^2\}}$	$-\infty < x < \infty$	$\beta > 0$
$N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$	$-\infty < x < \infty$	$\sigma^2 > 0$
Beta(α, β)	$\frac{1}{B(\alpha, \beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$\alpha > 0,\ \beta > 0$
Weibull(α, β)	$\beta\alpha x^{\alpha-1}\exp(-\beta x^\alpha)$	$x > 0$	$\alpha > 0,\ \beta > 0$
χ^2_k	$\frac{1}{2^{k/2}\Gamma(k/2)}x^{(k/2)-1}\exp(-\frac{1}{2}x)$	$x > 0$	$k = 1, 2, \ldots$
t_m	$\frac{\Gamma((m+1)/2)}{\Gamma(m/2)\sqrt{\pi m}}\ (1 + \frac{x^2}{m})^{-(m+1)/2}$	$-\infty < x < \infty$	$m = 1, 2, \ldots$
Pareto(θ)	$\frac{\theta}{x^{\theta+1}}$	$x > 1$	$\theta > 0$

Distribution	$E(X)$	$\text{var}\,(X)$	$G_X(z)$ or $M_X(t)$
Bernoulli(θ)	θ	$\theta(1-\theta)$	$1-\theta+\theta z$
(Discrete) Uniform(k)	$(k+1)/2$	$(k^2-1)/12$	$z(1-z^k)/\{k(1-z)\}$
Binomial(n,θ)	$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta z)^n$
Poisson(λ)	λ	λ	$\exp\{-\lambda(1-z)\}$
Geometric(θ)	$\frac{1}{\theta}$	$\frac{1-\theta}{\theta^2}$	$\frac{\theta z}{1-z(1-\theta)}$
NegativeBinomial(n,θ)	$\frac{n(1-\theta)}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta z}{1-z(1-\theta)}\right)^n$
Uniform(α,β)	$\frac{1}{2}(\alpha+\beta)$	$\frac{1}{12}(\beta-\alpha)^2$	$(e^{\beta t}-e^{\alpha t})/\{(\beta-\alpha)t\}$
Exponential(λ)	$1/\lambda$	$1/\lambda^2$	$\lambda/(\lambda-t)$
Gamma(ν,λ)	ν/λ	ν/λ^2	$\{\lambda/(\lambda-t)\}^\nu$
Cauchy	none	none	none
$N(\mu,\sigma^2)$	μ	σ^2	$\exp(\mu t+\frac{1}{2}\sigma^2 t^2)$
Beta(α,β)	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	${}_1F_1(\alpha;\beta;t)$
Weibull(α,β)	$\beta^{-1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right)$	$\beta^{-2/\alpha}\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\}$	none
χ^2_k	k	$2k$	$(1-2t)^{-k/2}$
t_m	0	$\frac{m}{m-2}$	none
Pareto(θ)	$\frac{\theta}{\theta-1}$	$\frac{\theta}{(\theta-1)^2(\theta-2)}$	none