Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M2S3 Statistical Theory I

Date: Friday, 28th May 2004 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

A formula sheet is given on pages 5 & 6.

1. (a) In a regular estimation problem T is an unbiased estimator of the single unknown parameter θ. State the relationship between the efficient score U_θ and T which provides necessary and sufficient conditions for the variance of T to attain the Cramer-Rao lower bound for θ.

(b) $Y_1, \ldots, Y_n$ are independent Poisson observations, each with unknown parameter θ.

(i) Show that

$$U_\theta = \frac{n(\overline{Y} - \theta)}{\theta}$$

where $\overline{Y} = \frac{1}{n} \sum Y_i$.

(ii) Is $\overline{Y}$ the minimum variance unbiased estimator of θ? Justify your answer.

(iii) Find the Fisher information with respect to ϕ, where $\phi = \theta(\theta + 2)$.

(iv) Show that there does not exist an unbiased estimator of ϕ whose variance attains the Cramer-Rao lower bound for ϕ.

(v) Find a random variable S, a function of $\overline{Y}$, which is unbiased for ϕ and show that

$$\text{var}S > \frac{4\theta(\theta + 1)^2}{n}.$$

2. (a) State (without proof)

(i) the Neyman Factorisation Theorem.

(ii) the Rao-Blackwell Theorem.

(b) Each of the two independent observations X and Y has an exponential distribution with unknown parameter θ.

(i) Show that $T = X + Y$ is a sufficient statistic for θ and state the distribution of T.

(ii) Given $x > 0$,

$$a(T) = \begin{cases} \dfrac{x}{T} & T > x, \\ 1 & \text{otherwise.} \end{cases}$$

Show that $a(T)$ is unbiased for $F_X(x)$, the cumulative distribution function of X evaluated at x.

(iii) It can be shown that the family of distributions of T is complete. Show that $a(T)$ is the unique function of T which is unbiased for $F_X(x)$.

(iv) Deduce that $a(T)$ is the minimum variance unbiased estimator of $F_X(x)$.

3. **(a)** In a statistical inference problem the maximum likelihood estimator of the unknown parameter θ is $\widehat{\theta}$. If $\lambda = g(\theta)$ is a one-one known function of θ show that $\widehat{\lambda} = g(\widehat{\theta})$ is a maximum likelihood estimator of λ

(b) The observation Y has a binomial distribution with known index n and unknown parameter θ $(0 < \theta < 1)$.

(i) Find the maximum likelihood estimator of θ and show that its variance is

$$\frac{\theta(1-\theta)}{n}.$$

(ii) By considering the random variable

$$g(Y) = \frac{1}{2}\left\{1 + (-1)^Y\right\}$$

and its expectation, or otherwise, show that the probability λ that Y is an even number is

$$\lambda = \frac{1}{2}\left\{1 + (1-2\theta)^n\right\}.$$

(iii) Find the maximum likelihood estimator $\widehat{\lambda}$ of λ.

4. **(a)** State and prove the Neyman-Pearson Lemma.

(b) The observations $y_1, \ldots, y_n$ form a random sample from the distribution with probability density function

$$2\theta y e^{-\theta y^2} \qquad (y > 0),$$

where $\theta > 0$ is unknown.

(i) Show that there is a uniformly most powerful test of size $\leq \gamma$ $(0 < \gamma < 1)$ of

$$H_0 : \theta = \frac{1}{2} \quad \text{against} \quad H_A : \theta > \frac{1}{2}.$$

(ii) Show that this uniformly most powerful test has critical region

$$R = \{(y_1, \ldots, y_n) : t(y_1, \ldots, y_n) < c\}$$

where $t(y_1, \ldots, y_n)$ is a function of the observations, to be determined, and

$$F(c) = \gamma,$$

where F is the cumulative distribution function of the chi-squared distribution on $2n$ degrees of freedom.

5. The observations $Y_1, \ldots, Y_n$ $(n > 2)$ are independent and each has a uniform distribution on $[\theta, 2\theta]$, where θ has a prior uniform distribution on $[0, a]$, where $a > 0$ is a known constant.

(a) Show that the posterior probability density of θ is $k\,\theta^{-n}$ $(b < \theta < c)$, where b and c are functions of the data to be determined, and

$$k = \frac{(n-1)(bc)^{n-1}}{c^{n-1} - b^{n-1}}.$$

(b) Show that if the loss incurred by estimating θ by $\widehat{\theta}$ is $(\theta - \widehat{\theta})^2$, then the Bayes Rule for estimating θ is given by

$$\widehat{\theta} = \frac{(n-1)}{(n-2)} \frac{(bc^{n-1} - b^{n-1}c)}{(c^{n-1} - b^{n-1})}.$$

(c) Show that the Bayes Rule for estimating θ when the loss incurred is now $\left|\theta - \widehat{\theta}\right|$ is given by

$$\widehat{\theta} = \left(\frac{2(bc)^{n-1}}{b^{n-1} + c^{n-1}} \right)^{\frac{1}{n-1}}.$$

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (given on Page 8), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma \mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2 \mathrm{Var}_{f_X}[X]$$

					ν
					(if $\alpha > 2$)
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta \in \mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$