UNIVERSITY OF LONDON

IMPERIAL COLLEGE LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2003

This paper is also taken for the relevant examination for the Associateship.

M2S3 STATISTICAL THEORY I

DATE: Wednesday, 28th May 2003 TIME: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical formula sheets are given on pages 6 & 7.

1. In a regular estimation problem, the joint distribution of the observations $Y_1, \ldots, Y_n$ depends on a single unknown parameter θ, and T is an estimator of θ.

a) Define *the efficient score*, U_θ, with respect to θ, and *the bias*, $b(\theta)$, of T.

b) Show that

$$E(U_\theta) = 0, \quad \text{and} \quad cov(U_\theta, T) = 1 + \frac{db(\theta)}{d\theta}.$$

c) In a particular case, there is a single observation from the normal distribution with mean 0 and variance θ^2, where $\theta > 0$, and

$$T = |Y|\,.$$

i) Show that

$$b(\theta) = \theta\left(\sqrt{\frac{2}{\pi}} - 1\right).$$

ii) Find U_θ and hence show that

$$cov\ (Y^2, |Y|) = \theta^3\sqrt{\frac{2}{\pi}}.$$

2. *a)* Explain what is meant by saying that a partition of the sample space is

i) *sufficient* for θ,

ii) *minimal sufficient* for θ,

where θ is the unknown parameter in a statistical inference problem.

b) Prove that a minimal sufficient partition of the sample space is unique.

c) If $Y_1, \ldots, Y_n$ form a random sample from the uniform distribution on $(\frac{1}{\theta}, \theta)$, where $\theta > 0$, find a minimal sufficient statistic for θ, stating clearly any results of which you make use.

3. *a)* State (without proof) the Rao-Blackwell Theorem.

b) The observations $Y_1, \ldots, Y_n$ are independent Poisson random variables each with unknown mean $\theta > 0$.

i) By considering the event $Y_1 = 2$, find an unbiased estimator of

$$\phi = \theta^2 e^{-\theta}.$$

ii) Show that $T = \sum_{i=1}^{n} Y_i$ is sufficient for θ, and that its family of distributions is complete.

iii) Hence find a minimum variance unbiased estimator of ϕ, stating clearly any results of which you make use.

4. *a)* Explain what is meant by *a pivotal quantity* for an unknown scalar parameter θ in an inference problem, and show how a pivotal quantity can be used to provide a confidence interval for θ.

b) The observations $Y_1, \ldots, Y_n$ form a random sample from the distribution with probability density function

$$f(y) = 2\theta y \exp(-\theta y^2), \qquad y \geq 0,$$

where $\theta > 0$.

i) Show that the maximum likelihood estimator of θ is

$$\widehat{\theta} = \frac{n}{\sum_{i=1}^{n} Y_i^2},$$

and find the asymptotic distribution of $\widehat{\theta}$ as $n \to \infty$ in this regular problem.

ii) Hence or otherwise show that for a suitable choice of the constant k (to be identified)

$$\frac{\widehat{\theta}}{1 \pm \frac{k}{\sqrt{n}}}$$

are approximate 95% confidence limits for θ for large n.

5. Each of n independent electrical components has a life-time which is exponentially distributed with parameter θ, where $\theta > 0$ has a prior distribution which is also exponential, but with known parameter $a > 0$.

Suppose that it is only possible to measure the time to failure, $Y_{(1)}$, of the component with the shortest life-time.

a) Show that, for $\theta > 0$, the likelihood function is $n\theta \exp(-n\theta y_{(1)})$ and identify the posterior distribution of θ.

b) Show that, under squared error loss, the Bayes Rule, $\widehat{\phi}$, for estimating ϕ, the expected life-time of a component, is given by $\widehat{\phi} = a + ny_{(1)}$.

c) Show that the posterior probability density function of ϕ is

$$\frac{\widehat{\phi}^2 \exp(-\widehat{\phi}/\phi)}{\phi^3}, \qquad \phi > 0,$$

and that the mode, ϕ_M, of this distribution is given by

$$\phi_M = \frac{\widehat{\phi}}{3}.$$

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\frac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\frac{1}{\theta}$	$\frac{(1-\theta)}{\theta^2}$	$\frac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\frac{n}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\frac{n(1-\theta)}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (given on Page 7), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma}$ $\quad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right)$ $\quad M_Y(t) = e^{\mu t}M_X(\sigma t)$ $\quad \mathrm{E}_{f_Y}[Y] = \mu + \sigma\mathrm{E}_{f_X}[X]$ $\quad \mathrm{Var}_{f_Y}[Y] = \sigma^2\mathrm{Var}_{f_X}[X]$

					ν (if $\alpha > 2$)
$Beta(\alpha, \beta)$	$(0, 1)$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$