BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M2S2

Statistical Modelling

Date: Wednesday, 21 May 2008 Time: 2 pm – 4 pm

Answer all questions. Each question carries equal weight.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) In a Bayesian framework, suppose you have observations $Y_1, \dots, Y_n$. Suppose that conditionally on θ they are independent and follow a Geometric(θ) distribution, i.e.

$$\mathrm{P}(Y_i = k|\theta) = \theta(1-\theta)^{k-1}, \quad k = 1, 2, \dots$$

Suppose that the a-priori distribution of θ is Beta(α, β) for known constants $\alpha, \beta > 0$, i.e. it has pdf

$$\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad \theta \in (0,1).$$

Find the posterior distribution for θ.

(ii) Find the projection matrix onto the space spanned by the vectors $(1,1,1)^T$ and $(0,2,0)^T$.

(iii) In a linear model with second order assumptions, define the vector of residuals. Derive the variances of its components.

(iv) In a linear model with normal theory assumptions, for $0 < \alpha < 1$, construct a $1-\alpha$ confidence interval for the standard deviation σ of the errors.

2. Let Y_{ij}, $i = 1, \dots, m, j = 1, \dots n$, be independent random variables.
Suppose $Y_{ij} \sim$ Poisson(λ_i), i.e.

$$\mathrm{P}(Y_{ij} = k) = \frac{\lambda_i^k}{k!}e^{-\lambda_i} \text{ for } k = 0, 1, 2, \dots$$

where $\lambda_i > 0$ are unknown.
Consider the following hypotheses:

$$H_0 : \lambda_1 = \dots = \lambda_m, \quad H_1 : \text{ not } H_0$$

(i) Derive the maximum likelihood estimator of $\lambda_1, \dots, \lambda_m$ under H_1.

(ii) Derive the maximum likelihood estimator of λ_1 under H_0.

(iii) Find (and simplify) the likelihood ratio test statistic T for the above hypotheses.

(iv) What is the asymptotic distribution of T under H_0 as $n \to \infty$?
Describe how you would construct an asymptotic test for H_0 against H_1.

3. (i) State the second order assumptions and the normal theory assumptions in a linear model.

   (ii) Consider a linear model with full rank.

   (a) State the formula for the least squares estimator.

   (b) Show that the least squares estimator is unbiased.

   (c) Under second order assumptions, derive the covariance matrix of the least squares estimator.

   (iii) Define a consistent sequence of estimators.
   Give a sufficient condition for consistency based on means and variances.

   (iv) Consider the simple linear regression

$$\mathrm{E}\,Y_i = \beta_1 + \beta_2 x_i, \quad i = 1, \ldots, n.$$

   (a) Write the model in matrix notation.

   (b) Compute the covariance matrix of the least squares estimators of β_1 and β_2.

   (c) Suppose the x_i are observations of independent and identically distributed random variables and the linear regression model is conditional on the x_i. Under what conditions are the components of the least squares estimator consistent?

4. (i) Carefully define the central F-distribution.

   (ii) State the Fisher-Cochran Theorem.

   (iii) In the linear model $\mathrm{E}\,\boldsymbol{Y} = \underline{\boldsymbol{X}}\boldsymbol{\beta}$ with normal theory assumptions, describe the F-test for

$$H_0 : \mathrm{E}\,\boldsymbol{Y} = \underline{\boldsymbol{X}}_0\boldsymbol{\beta}_0 \text{ for some } \boldsymbol{\beta}_0$$

   against

$$H_1 : \mathrm{E}\,\boldsymbol{Y} \neq \underline{\boldsymbol{X}}_0\boldsymbol{\beta}_0 \text{ for all } \boldsymbol{\beta}_0,$$

   where $\underline{\boldsymbol{X}}_0$ is a known matrix such that its columns are elements of the space spanned by the columns of the design matrix $\underline{\boldsymbol{X}}$.
   Clearly state the test statistic, its distribution and the decision rule.

   (iv) Starting from the Fisher-Cochran Theorem, derive the distribution of the test statistic for the F-test both under the null hypothesis and the alternative hypothesis.
   You may use results about projection matrices from lectures without proof.