1. Let $\boldsymbol{Y} = [Y_1, \ldots, Y_n]^\top$ denote a random sample from distribution

$$f_Y(y) = \begin{cases} (\theta+2)\, y^{\theta+1} & \text{for} \quad y \in [0,1] \\ 0 & \text{for} \quad y \notin [0,1] \end{cases}$$

where $\theta > -2$.

(a) Write down the likelihood of $\boldsymbol{Y}$.

(b) Find the maximum likelihood estimator of $\theta$, $\widehat{\theta}$.

(c) Write down the invariance property of maximum likelihood estimators.

(d) Find the maximum likelihood estimator of $\tau = \log(\theta + 1)$.

2. Let $\boldsymbol{X} = [X_1, \ldots, X_n]^\top$ denote a random sample from distribution

$$f_X(x) = \begin{cases} \left(\frac{3}{\theta^3}\right)(\theta - x)^2 & \text{for} \quad 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the population mean of this distribution.

(b) Describe the method of moments procedure, in general terms.

(c) Find the method of moments estimator of $\theta$, for the distribution given here.

(d) Find the variance of this estimator.

3. Let $\boldsymbol{U} = [U_1, \ldots, U_n]^\top$ denote a random sample from a Bernoulli distribution with probability of success $\eta$.

(a) Write down the likelihood of $\boldsymbol{U}$.

(b) Say we assign a prior distribution of $\eta$ to be given by

$$p(\eta) = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \eta^{\alpha-1} (1-\eta)^{\alpha-1}, \ \eta \in (0,1),$$

where $\alpha > 0$ is specified.

(i) Find the posterior distribution of $\eta$, given we have observed the random sample $\boldsymbol{U}$.

(ii) Find the posterior mode of the distribution. For simplicity restrict your solution to the cases $\sum_{i=1}^n u_i \neq 0$ or $n$.

(iii) Under what loss function is the posterior mode the minimum expected posterior loss estimate of $\eta$?

(c) The prior assigned is a special case of the Beta distribution given in general for a random variable $Q$ as

$$p(q) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha-1} (1-q)^{\beta-1}, \ q \in (0,1).$$

Plot a rough scetch of this prior for $\beta = \frac{3}{4}\alpha$, $\beta = \alpha$ and $\beta = 2\alpha$ when $\alpha = 2$.
What prior belief in this case have we incorporated in taking $\alpha = \beta$ in the prior for $\eta$ in part (b)?

4. The life-time of a lightbulb $V$ is modelled as following the distribution with density

$$f_V(v) = \theta \exp(-\theta v), \; v > 0.$$

A manufacturing company claims that they make lightbulbs such that $\theta = 1$, and a consumer-interest organisation wishes to test whether this is true. A random sample of size $n$, $\boldsymbol{V} = [V_1, \ldots, V_n]^T$, is collected to test the hypothesis.

(a) Using moment generating functions (or otherwise) identify the distribution of

$$S_1 = \sum_{i=1}^{n} V_i.$$

(b) Verify that the distribution of $S_2 = 2\theta S_1$ is $\chi^2_{2n}$.

(c) We wish to test the hypothesis

$$H_0 : \; \theta = 1,$$

versus

$$H_1 : \; \theta > 1,$$

based on $S_1$, at level significance level $\alpha$. Describe how to perform this statistical test.

(d) Say we observe $s_1 = 9.891$, when $n = 10$. Can we reject the null hypothesis at the $5\%$ level, given that

$$\begin{aligned} \chi^2_{10,0.025} &= 3.2470, & \chi^2_{10,0.05} &= 3.9403, & \chi^2_{10,0.1} &= 4.8652, \\ \chi^2_{20,0.025} &= 9.5908, & \chi^2_{20,0.05} &= 10.8508, & \chi^2_{20,0.1} &= 12.4426, \end{aligned}$$

where $\chi^2_{k,\alpha}$ is the $\alpha$th percentile of the $\chi^2_k$ distribution.

5. A drug trial is conducted where the three first patients are given drug $A$, the next three patients are given drug $B$ and the final three patients are given no drug at all. After two weeks the blood-pressure of patient $i$ is recorded as $Y_i$. It is assumed that $Y_i \sim N(\mu_i, \sigma^2),\ i = 1, \ldots, 9$, independently, and that the mean response of patient $i$ is $\mu_A$ if they were administered drug $A$, $\mu_B$ if they were administered drug $B$, and $\mu_C$ if they were given no drug at all.

(a) Write down the design matrix $\boldsymbol{X}$ for this set-up.

(b) Write down the linear model for $\boldsymbol{Y}$ in terms of $\boldsymbol{X}$, $\boldsymbol{\beta} = [\mu_A\ \mu_B\ \mu_C]^T$ and random error vector $\boldsymbol{\epsilon}$. Write down the distribution of $\boldsymbol{\epsilon}$.

(c) Write down the Gauss-Markov theorem.

(d) Given that $\boldsymbol{y} = [y_1, \ldots, y_9]^T$ was observed, estimate the parameter vector $\boldsymbol{\beta}$ such that it is optimal in the class of estimators encompassed by the Gauss-Markov theorem.

(e) Find $\mathrm{Var}\left(\widehat{\boldsymbol{\beta}}\right)$. Are the estimates of $\mu_A$ and $\mu_B$ independent? Please give your reasoning carefully.

**DISCRETE DISTRIBUTIONS**

| | RANGE $\mathbb{X}$ | PARAMETERS | MASS FUNCTION $f_X$ | CDF $F_X$ | $\mathrm{E}_{f_X}[X]$ | $\mathrm{Var}_{f_X}[X]$ | MGF $M_X$ |
|---|---|---|---|---|---|---|---|
| $Bernoulli(\theta)$ | $\{0,1\}$ | $\theta \in (0,1)$ | $\theta^x(1-\theta)^{1-x}$ | | $\theta$ | $\theta(1-\theta)$ | $1-\theta+\theta e^t$ |
| $Binomial(n,\theta)$ | $\{0,1,...,n\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ | | $n\theta$ | $n\theta(1-\theta)$ | $(1-\theta+\theta e^t)^n$ |
| $Poisson(\lambda)$ | $\{0,1,2,...\}$ | $\lambda \in \mathbb{R}^+$ | $\dfrac{e^{-\lambda}\lambda^x}{x!}$ | | $\lambda$ | $\lambda$ | $\exp\{\lambda(e^t-1)\}$ |
| $Geometric(\theta)$ | $\{1,2,...\}$ | $\theta \in (0,1)$ | $(1-\theta)^{x-1}\theta$ | $1-(1-\theta)^x$ | $\dfrac{1}{\theta}$ | $\dfrac{(1-\theta)}{\theta^2}$ | $\dfrac{\theta e^t}{1-e^t(1-\theta)}$ |
| $NegBinomial(n,\theta)$ | $\{n,n+1,...\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$ | | $\dfrac{n}{\theta}$ | $\dfrac{n(1-\theta)}{\theta^2}$ | $\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$ |
| or | $\{0,1,2,...\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{n+x-1}{x}\theta^n(1-\theta)^x$ | | $\dfrac{n(1-\theta)}{\theta}$ | $\dfrac{n(1-\theta)}{\theta^2}$ | $\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$ |

For **CONTINUOUS** distributions (given on Page 7), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma \mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2 \mathrm{Var}_{f_X}[X]$$

## CONTINUOUS DISTRIBUTIONS

| | RANGE $\mathbb{X}$ | PARAMETERS | PDF $f_X$ | CDF $F_X$ | $\mathrm{E}_{f_X}[X]$ | $\mathrm{Var}_{f_X}[X]$ | MGF $M_X$ |
|---|---|---|---|---|---|---|---|
| $Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$) | $(\alpha,\beta)$ | $\alpha<\beta\in\mathbb{R}$ | $\dfrac{1}{\beta-\alpha}$ | $\dfrac{x-\alpha}{\beta-\alpha}$ | $\dfrac{(\alpha+\beta)}{2}$ | $\dfrac{(\beta-\alpha)^2}{12}$ | $\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$ |
| $Exponential(\lambda)$ (standard model $\lambda=1$) | $\mathbb{R}^+$ | $\lambda\in\mathbb{R}^+$ | $\lambda e^{-\lambda x}$ | $1-e^{-\lambda x}$ | $\dfrac{1}{\lambda}$ | $\dfrac{1}{\lambda^2}$ | $\left(\dfrac{\lambda}{\lambda-t}\right)$ |
| $Gamma(\alpha,\beta)$ (standard model $\beta=1$) | $\mathbb{R}^+$ | $\alpha,\beta\in\mathbb{R}^+$ | $\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ | | $\dfrac{\alpha}{\beta}$ | $\dfrac{\alpha}{\beta^2}$ | $\left(\dfrac{\beta}{\beta-t}\right)^\alpha$ |
| $Weibull(\alpha,\beta)$ (standard model $\beta=1$) | $\mathbb{R}^+$ | $\alpha,\beta\in\mathbb{R}^+$ | $\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$ | $1-e^{-\beta x^\alpha}$ | $\dfrac{\Gamma\left(1+\alpha^{-1}\right)}{\beta^{1/\alpha}}$ | $\dfrac{\Gamma\left(1+2\alpha^{-1}\right)-\Gamma\left(1+\alpha^{-1}\right)^2}{\beta^{2/\alpha}}$ | |
| $Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$) | $\mathbb{R}$ | $\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$ | $\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$ | | $\mu$ | $\sigma^2$ | $\exp\left\{\mu t+\dfrac{\sigma^2t^2}{2}\right\}$ |
| $Student(\nu)$ | $\mathbb{R}$ | $\nu\in\mathbb{R}^+$ | $\dfrac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\sqrt{\pi\nu}\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$ | | $0$ (if $\nu>1$) | $\dfrac{\nu}{\nu-2}$ (if $\nu>2$) | |
| $Pareto(\theta,\alpha)$ | $\mathbb{R}^+$ | $\theta,\alpha\in\mathbb{R}^+$ | $\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$ | $1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$ | $\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$) | $\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$) | |
| $Beta(\alpha,\beta)$ | $(0,1)$ | $\alpha,\beta\in\mathbb{R}^+$ | $\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | | $\dfrac{\alpha}{\alpha+\beta}$ | $\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | |