Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M2S2 Linear Modelling

Date: Tuesday, 25th May 2004 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are included on pages 5 & 6.

1. Let $(X_1, \ldots, X_n)$ denote a random sample from the distribution

$$f_X(x) = \begin{cases} 3e^{-3(x-2\theta)} & \text{for} \quad x > 2\theta \\ 0 & \text{for} \quad x \leq 2\theta \end{cases}$$

(a) Find $\widehat{\theta}_X$, the Method of Moments estimator of θ.

(b) Find the expectation of $\widehat{\theta}_X$.

(c) Find the variance of $\widehat{\theta}_X$.

(d) Let $(Y_1, \ldots, Y_m)$ denote a random sample from the distribution

$$f_Y(y) = \begin{cases} e^{-(y-2\theta)} & \text{for} \quad y > 2\theta \\ 0 & \text{for} \quad y \leq 2\theta \end{cases}$$

Denote the Method of Moments estimator of θ from this sample $\widehat{\theta}_Y$. Assume that the cost of collecting a sample of size m from f_Y is equal to that of collecting one of size n from f_X. For what range of m and n is $\widehat{\theta}_X$ to be preferred to $\widehat{\theta}_Y$ (considering their respective mean squared error)?

2. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ denote a random sample of size n from

$$f_X(x) = \frac{\mu^x e^{-\mu}}{x!}, \; x = 0, 1, 2, 3, \ldots$$

(a) Write down the likelihood of $\boldsymbol{X}$.

(b) Find $\widehat{\mu}$, the Maximum Likelihood Estimator of μ.

(c) Let $\tau_0(\mu) = \mathsf{P}(X_i = 0)$, the probability that an observation from distribution $f_X(x)$ is equal to 0. Find $\tau_0(\mu)$ as a function of μ.

(d) Find the Maximum Likelihood Estimator of $\tau_0(\mu)$, stating any properties about Maximum Likelihood Estimators you use, and checking their applicability.

(e) Let $\tau_1(\mu)$ denote the probability that an observation from distribution $f_X(x)$ is equal to 1. Why cannot the result stated in **(d)** be used in this instance to find a Maximum Likelihood Estimator of $\tau_1(\mu)$?
[Hint: Make a rough plot of $\tau_1(\mu)$ versus μ and comment].

3. Consider a random sample $\boldsymbol{X} = (X_1, \ldots, X_n)$ of size n from a normal population with mean $\mu_1 + \mu_2$ and known variance σ_0^2, and a random sample $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ of size m from a normal population with mean $\mu_1 - \mu_2$ and the same known variance σ_0^2.
Assume that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent.

(a) For an arbitrary random variable Z with density $f_Z(z;\theta)$, write down the definition of a pivotal quantity Q based on a random sample $(Z_1, \ldots, Z_n)$ from f_Z.

(b) Find a pivotal quantity for $\mu_1 + \mu_2$, based on the sample $\boldsymbol{X}$.

(c) Similarly, find a pivotal quantity for $\mu_1 - \mu_2$, based on the sample $\boldsymbol{Y}$.

(d) Construct a $100(1-\alpha)\%$ confidence interval for μ_1, defining any normal percentiles appropriately.

(e) Describe how to test the hypothesis

$$H_0 : \mu_1 = \mu_0$$

versus the alternative

$$H_1 : \mu_1 \neq \mu_0$$

at size α.

4. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ denote a random sample of size n from

$$f_{X|\mu}(x \mid \mu) = \frac{\mu^x e^{-\mu}}{x!} \qquad x = 0, 1, 2, 3, \ldots$$

Assume a prior distribution on μ as

$$p_\mu(\mu) = \frac{1}{\Gamma(\alpha+1)} \mu^\alpha e^{-\mu}, \ \mu > 0.$$

(a) Find the joint distribution of μ and $\boldsymbol{X}$.

(b) Find the posterior distribution of μ.

(c) Find a best point estimate of μ under square-error loss.

(d) Describe how to find an interval estimate of μ, defining the percentiles of the posterior distribution appropriately.

5. A sample of independent observations, $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ of size n, sampled at fixed positive design points $(x_1, \ldots, x_n)$, is modelled with means and variances

$$\begin{aligned} \mathsf{E}\,(Y_i) &= \beta_1^*\,(\sqrt{x_i} + \beta_2^*) \\ \mathsf{var}\,(Y_i) &= \sigma^2 \end{aligned}$$

(a) Restate this as a linear model in $\beta_0 = \beta_1^* \beta_2^*$ and $\beta_1 = \beta_1^*$, and specify the associated design matrix. [Hint: it may be easier to work with $u_i = \sqrt{x_i}$].

(b) Write down the theorem which ensures that the least-squares estimator has minimum variance in a certain class of estimators, noting carefully any necessary assumptions on $\boldsymbol{Y}$.

(c) Find the least-squares estimator of $\boldsymbol{\beta} = [\beta_0, \beta_1]^T$.

(d) Find the covariance matrix of $\boldsymbol{\beta}$.

(e) Let $\widehat{\mu}(x_0)$ be the estimator $\beta_1^* \left(\sqrt{x_0} + \beta_2^*\right)$ based on the least-squares estimator of $\boldsymbol{\beta}$.

(i) Find $\mathsf{var}\,(\widehat{\mu}(x_0))$.

(ii) Minimize this as a function of x_0, and calculate x^*, the minimizing value of x_0.

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (given on Page 8), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma \mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2 \mathrm{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\frac{1}{\beta-\alpha}$	$\frac{x-\alpha}{\beta-\alpha}$	$\frac{(\alpha+\beta)}{2}$	$\frac{(\beta-\alpha)^2}{12}$	$\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2\alpha^{-1})-\Gamma(1+\alpha^{-1})^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$\exp\left\{\mu t+\frac{\sigma^2t^2}{2}\right\}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\frac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\frac{\theta}{\theta+x}\right)^\alpha$	$\frac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	